PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Magnetic Reconnection at Three-Dimensional Null Points

E. R. Priest and V. S. Titov

Phil. Trans. R. Soc. Lond. A 1996 **354**, 2951-2992
doi: 10.1098/rsta.1996.0136

Magnetic reconnection at three-dimensional null points

By E. R. Priest and V. S. Titov†

Department of Mathematical and Computational Sciences, The University, St Andrews KY16 9SS, UK

Contents

The skeleton of an isolated null point in three dimensions consists of a 'spine curve' and a 'fan surface'. Two isolated magnetic field lines approach (or recede from) the null point from both directions along the spine, and a continuum of field lines recedes from (or approaches) the null in the plane of the fan surface. Two bundles of field lines approach the null point around the spine (one from each direction) and spread out near the fan. The kinematics of steady reconnection at such a null point is considered, depending on the nature of the imposed boundary conditions on the surface that encloses the null, in particular on a cylindrical surface with its axis along the spine. Three kinds of reconnection are discovered. In 'spine reconnection' continuous footpoint motions are imposed on the curved cylindrical surface, crossing the fan and driving singular jetting flow along the spine. In 'fan reconnection' continuous footpoint motions are prescribed on the ends of the cylinder, crossing the spine and driving a singular swirling motion at the fan. An antireconnection theorem is proved, which states that steady MHD reconnection in three dimensions with plasma flow across the spine or fan is impossible in an inviscid plasma with a highly subAlfvenic flow and uniform magnetic diffusivity. One implication of this is

† Permanent address: Program Systems Institute of the Russian Academy of Sciences, Pereslavl-Zalessky 152140, Russia.

Phil. Trans. R. Soc. Lond. A (1996) **354**, 2951–2992

that reconnection tends to be an inherently nonlinear phenomenon. A linear theory for slow steady reconnection is developed which demonstrates explicitly the nature of the spine singularity in spine reconnection. Finally, the properties of 'separator reconnection' in complex configurations containing two null points are discussed by means of analytical examples.

1. Introduction

The problem of the breaking and reconnection of magnetic field lines in two dimensions has been widely studied over the past 20 years and is now fairly well understood, although some key questions about the details remain (Scholer 1991). The original fast reconnection mechanism of Petschek (1964) has been generalized in two ways to give a new generation of reconnection models. These include: families of almost-uniform reconnection models (Priest & Forbes 1986; Jardine & Priest 1988, 1989, 1990), in which the inflow boundary velocity at large distances has a range of inclinations; and families of non-uniform reconnection models (Priest & Lee 1990; Strachan and Priest 1994), in which the inflow magnetic field is highly curved. Furthermore, the remarkable numerical experiments of Biskamp (1986) show features such as separatrix jets and reversed current spikes at the ends of the central diffusion region, and the scalings of his particular set of experiments have been reproduced with the analytical models by Priest & Forbes (1992). In addition, numerical studies of Yan *et al.* (1992, 1993) have, by adopting the appropriate boundary conditions, reproduced the almost-uniform and non-uniform reconnection models.

The complex three-dimensional aspects of reconnection have just begun to be understood. Bulanov *et al.* (1984, 1992) have obtained self-similar solutions near a null point. Schindler *et al.* (1988) and Hesse & Schindler (1988) have proposed a generalization of reconnection to include all effects of local non-idealness that produce an electric field component parallel to the magnetic field, including for example double layers. Here we focus on one particular family of three-dimensional reconnection models, namely those that can occur near a three-dimensional null point. In a companion paper, we discuss another family, namely three-dimensional reconnection in the absence of null points, at so-called quasi-separatrix layers (Priest & Démoulin 1995). Lau & Finn (1990) follow Greene (1988) in restricting the term reconnection to phenomena arising in resistive MHD associated with boundary layers and isolated null points. They introduce the concepts of *null-null lines*, which are magnetic field lines linking null points; γ-*lines* which are isolated field lines that start from or end at a null point; Σ-*surfaces* which are magnetic flux surfaces containing field lines that all connect to a null point. They also consider kinematic reconnection near a single null point and a pair of null points, in which the magnetic field is prescribed and the electric potential (and therefore the velocity of transport of field lines) is deduced.

Priest & Forbes (1989) suggested that three-dimensional reconnection may be defined to occur when there is an electric field ($E_{||}$) parallel to field lines (known as *potential singular lines*) which are potential reconnection locations and near which the magnetic field has an X-type topology in a plane normal to that field line. In general there is a continuum of neighbouring potential singular lines, and which one supports reconnection depends on the imposed flow or electric field. Priest & Forbes (1992) describe a process of *magnetic flipping*, in which reconnection takes place even in the absence of magnetic nulls near a singular field line. A more detailed mathematical analysis of the flipping layers is given by Priest *et al.* (1994). In the present paper we focus on other types of three-dimensional magnetic reconnection, namely those that can ocur near magnetic nulls. As a background and contrast to the present work, § 2 discusses the motion of magnetic field lines in general and the application to two-dimensional X-points and two-and-a-half-dimensional configurations in particular. Section 3 describes the nature of three-dimensional null points, and then § 4 discusses two types of reconnection that may take place near them, called *spine* reconnection and *fan* reconnection. Section 5 proves an antireconnection theorem and gives a general solution for the linear resistive MHD equations describing slow steady flows of plasma near a radial null point of a magnetic field. Also the implications of such a theorem and another proof based on the general solution of the corresponding boundary-value problem for spine reconnection are given together with a discussion of the qualitative features of a nonlinear model. Finally, § 6 extends the ideas to a pair of null points and § 7 presents the conclusions.

2. Motion of magnetic field lines

We consider ideal steady MHD flow satisfying

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \boldsymbol{0} \tag{2.1}$$

and

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}. \tag{2.2}$$

Thus from (2.2) the electric field may be written in terms of an electric potential (Φ) as

$$\boldsymbol{E} = -\nabla \Phi. \tag{2.3}$$

The component of (2.1) parallel to $\boldsymbol{B}$, obtained by taking the scalar product with $\boldsymbol{B}$, gives

$$\boldsymbol{B} \cdot \nabla \Phi = 0, \tag{2.4}$$

so that Φ is constant along magnetic field lines. On the other hand, the component of (2.1) perpendicular to $\boldsymbol{B}$ may be found by taking the vector product with $\boldsymbol{B}$ and gives the component ($\boldsymbol{v}_\perp$) of $\boldsymbol{v}$ normal to $\boldsymbol{B}$ as

$$\boldsymbol{v}_\perp = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}. \tag{2.5}$$

Finally, the component ($\boldsymbol{v}_{||}$) along the field can be determined in, for example, the case of incompressible flow from

$$\nabla \cdot \boldsymbol{v} = 0. \tag{2.6}$$

Thus, if $\boldsymbol{v}_\perp$ (and therefore Φ) is specified at one end of each field line, equation (2.4) determines Φ throughout the whole volume threaded by such field lines. Then (2.3) determines $\boldsymbol{E}$ and (2.5) $\boldsymbol{v}_\perp$ everywhere. If $\boldsymbol{v}_\perp$ is singular somewhere, then non-ideal effects such as magnetic diffusion are needed there: this may occur either when $\boldsymbol{E}$ is singular or when $\boldsymbol{B}$ vanishes (i.e. at null points).

(a) Two-dimensional X-point

Consider the simple potential X-point in two dimensions with field components

$$B_x = x, \quad B_y = -y. \tag{2.7}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

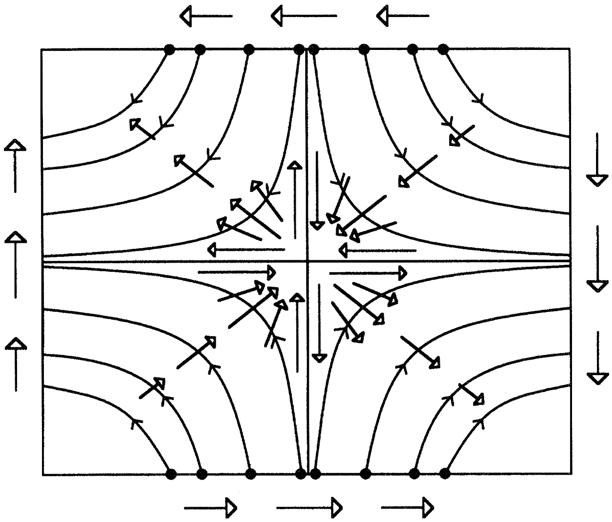

Figure 1. Field lines and flow vectors (solid arrows) for a two-dimensional X-point.

If the flow is also two-dimensional with components (v_x, v_y) that depend on x and y alone, then (2.1) implies that the electric field

$$\boldsymbol{E} = E(x, y)\hat{\boldsymbol{z}}$$

has only a z-component, whereupon (2.2) or (2.3) imply that

$$E = E_0 \tag{2.8}$$

is uniform in space. In other words (2.4) is satisfied identically and so we do not need to follow the general procedure of mapping Φ along field lines (i.e. solving (2.4)) in order to determine Φ and therefore $\boldsymbol{E}$.

Having found $\boldsymbol{E}$, (2.5) determines the components of $\boldsymbol{v}_\perp$ as

$$v_{\perp x} = -\frac{E_0 y}{x^2 + y^2}, \qquad v_{\perp y} = -\frac{E_0 x}{x^2 + y^2}, \tag{2.9}$$

so that the magnitude of $\boldsymbol{v}_\perp$ is

$$v_\perp = \frac{E}{B} = \frac{E_0}{(x^2 + y^2)^{1/2}}. \tag{2.10}$$

It can be seen that for $E_0 > 0$ the flow is directed inwards in the first and third quadrants and outwards in the second and fourth, so that it carries in oppositely directed field lines, reconnects them and carries them out in the classical manner. Also, at the origin there is a singularity where the field goes to zero and the flow becomes infinite. This is effectively because the field lines are being stretched indefinitely as they approach the neutral point. As the field lines approach the origin, so $v_\perp$ increases like r^{-1}.

Now if we assume incompressibility $v_{||}$ may be deduced from (2.6). The resulting components of the total velocity are (when $v_x = v_y$ on $y = x$)

$$v_x = -\frac{E_0}{2y}, \quad v_y = -\frac{E_0}{2x}, \tag{2.11}$$

from which we can again see a singularity at the origin. However, there is a new feature, namely that $v_{||}$ is singular all along the separatrices $x = 0$ and $y = 0$. This is a plasma effect rather than a field line motion effect, namely that the plasma elements are having to expand indefinitely all along the separatrices.

(*b*) *Magnetic flipping*

Consider next the two-and-a-half-dimensional field

$$B_x = x, \quad B_y = -y, \quad B_z = b_0 \tag{2.12}$$

considered by Priest & Forbes (1989), where b_0 is constant. As before, the x- and y-components of (2.2) imply that the z-component of the electric field

$$E_z = E_0$$

is constant, but now the z-component gives one relation between the x- and y-components, namely

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0. \tag{2.13}$$

A second relation arises from (2.4), namely

$$\boldsymbol{B} \cdot \boldsymbol{E} = xE_x - yE_y + b_0 E_0 = 0. \tag{2.14}$$

Eliminating E_y between (2.13) and (2.14) gives a single equation for E_x which may be solved by the method of characteristics to give

$$E_x = -\frac{E_0 b_0}{2x} + yF,$$

where $F = F(xy)$ is an arbitrary function of xy, constant along field lines. Then E_y follows from (2.14) as

$$E_y = \frac{E_0 b_0}{2y} + xF.$$

Having determined the electric field, the plasma flow normal to the magnetic field follows from (2.5) as

$$\boldsymbol{v}_\perp = \frac{b_0}{x^2 + y^2 + b_0^2} \times \left(\frac{E}{2yb_0}(2y^2 + b_0^2) + xF, \frac{E_0}{2xb_0}(2x^2 + b_0^2) - yF, E_0 \frac{y^2 - x^2}{2xy} - \frac{x^2 + y^2}{b_0} F \right). \tag{2.15}$$

The interesting new feature is that, as well as $v_{\perp x}$ and $v_{\perp y}$ in general becoming singular on the z-axis ($x = y = 0$), $v_{\perp x}$ also becomes singular on the $y = 0$ plane and $v_{\perp y}$ on the $x = 0$ plane; also $v_{\perp z}$ becomes singular on both planes. The physical cause is that the field lines become stretched indefinitely as they approach the z-axis.

If the extra condition that $\nabla \cdot \boldsymbol{v} = 0$ is imposed, then the flow components become

$$v_x = \frac{E_0}{2y} - \frac{x}{b_0} G, \qquad v_y = \frac{E_0}{2x} + \frac{y}{b_0} G, \qquad v_z = -F - G,$$

where $G = G(xy)$ is another arbitrary function that is constant on field lines.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

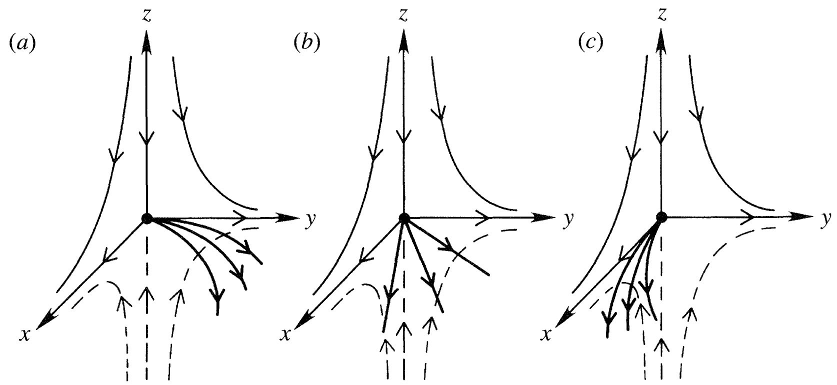

Figure 2. Field lines near a three-dimensional null point with (*a*) $0 < b < 1$, (*b*) $b = 1$, (*c*) $b > 1$.

3. Three-dimensional null points

Consider the simplest type of current-free null point at the origin, namely that with field components

$$B_x = x, \quad B_y = by, \quad B_z = -(b+1)z \tag{3.1}$$

satisfying $\nabla \cdot \boldsymbol{B} = 0$ and $\boldsymbol{j} = \nabla \times \boldsymbol{B}/\mu = 0$, where b is a positive constant and we have set the magnetic field scale (B_e) at unit distance along the x-axis equal to unity. The field lines are given by

$$\frac{dx}{x} = \frac{dy}{by} = -\frac{dz}{(b+1)z},$$

or

$$y = c_1 x^b, \quad z = c_2 x^{-(b+1)}, \tag{3.2}$$

where c_1 and c_2 are constants.

The field lines which link to or emanate from the null point are of two distinct types. There is one very special field line, here the z-axis, which is isolated in the sense that no neighbouring field lines pass through the null point. We refer to it as the *null-spine*, and it is the same as Lau & Finn's 'γ-line' or 'one-dimensional manifold'. The spine passes right through the null point and the magnetic field of the spine is directed either towards or away on both sides: thus in figure 2 the field is negative on the positive z-axis and positive on the negative (dashed) z-axis. All of the remaining field lines which link to the null form a surface, here the xy-plane, and fan out from the null in the surface. We refer to this surface as the *null fan*: it is the same as Lau & Finn's 'Σ-surface' or 'two-dimensional manifold'.

The field lines in the null fan behave differently depending on the value of b. If $0 < b < 1$, all except the x-axis itself emanate from the null point in a direction parallel to the y-axis and then curve away from the y-axis. If $b = 1$, all the field lines are straight lines. If $b > 1$ all except the y-axis emanate from the null point in a direction parallel to the x-axis and then curve away from the x-axis.

Field lines which are near to the null point but don't pass through it form a bundle around the spine which approach the null from both sides and then spread out above and below the fan surface. In the present example the field lines of the spine approach the null, while those of the fan radiate out of the null. Such a null we refer to as a *positive null point*, since most of the field lines passing through the null (i.e. those in the fan) emanate from it. For a *negative null point* (such as obtained by replacing $\boldsymbol{B}$ by $-\boldsymbol{B}$ in (3.1)), most of the field lines approach the null: the field lines of the spine both recede from the null, while all those of the fan approach it. Null points for which the field lines of the fan spiral into or away from the null (see below) are called *spiral null points*, while those for which the field lines of the fan approach or recede from it from radial directions are called *radial null points*. Radial nulls for which the field lines approach or recede from all directions are called *proper* (in our case when $b = 1$), while those which do so from essentially only two directions are called *improper* (here $b \neq 1$), following the standard nomenclature for phase plane critical points.

Fukao *et al.* (1975) have studied more general null points than (3.1) (see also Parnell *et al.* 1996), but the key features of possessing a null-spine and a null-fan are also present in such null points. Close to a null at the origin, say, the magnetic field is of the form

$$\boldsymbol{B} = \boldsymbol{M} \cdot \boldsymbol{r},$$

where $\boldsymbol{M}$ is a matrix with elements $\boldsymbol{M}_{ij} = \partial B_i/\partial x_j$. The equation $\nabla \cdot \boldsymbol{B} = 0$ implies that the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ of $\boldsymbol{M}$ sum to zero

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

The case when one eigenvalue vanishes is structurally unstable in the sense that it is not generally preserved when the field is perturbed; it gives a line of X-points when the other two eigenvalues are real and of 0-points when they are imaginary. When all three eigenvalues are real, we have a radial null point: examples of this are when the magnetic field is potential, force-free or magnetostatic; when the current vanishes at the null, $\boldsymbol{M}$ is symmetric and the principal axes are orthogonal so that the fan is perpendicular to the spine. When one eigenvalue is real and the others are complex conjugates, $\boldsymbol{M}$ is asymmetric and we have a spiral null point with a current flowing through it; the eigenvector corresponding to the real eigenvalue is directed along the spine; when the current in the fan vanishes, the fan is perpendicular to the spine.

4. The nature of reconnection at a three-dimensional null point

(a) Spine reconnection

Having described above the nature of the magnetic field near a three-dimensional null point (see also Cowley (1973), Stern (1973), Fukao (1975), Greene (1988), Lau & Finn (1990) for more details), let us now proceed to investigate the nature of reconnection near such a point. In doing so, we seek to determine what forms of Φ are allowable from (2.4) (and therefore $\boldsymbol{E}$ and $\boldsymbol{v}_\perp$ from (2.3) and (2.5)) and to find how magnetic flux surfaces reconnect. As in two dimensions, we shall find that the nature and speed of the reconnection depend on the motion of the flux sources and therefore of the footpoints on some given boundary.

For simplicity we start with a radial null point having $b = 1$, for which the field components are

$$B_x = x, \quad B_y = y, \quad B_z = -2z \tag{4.1}$$

or, in cylindrical polars,

$$B_R = R, \quad B_\phi = 0, \quad B_z = -2z. \tag{4.2}$$

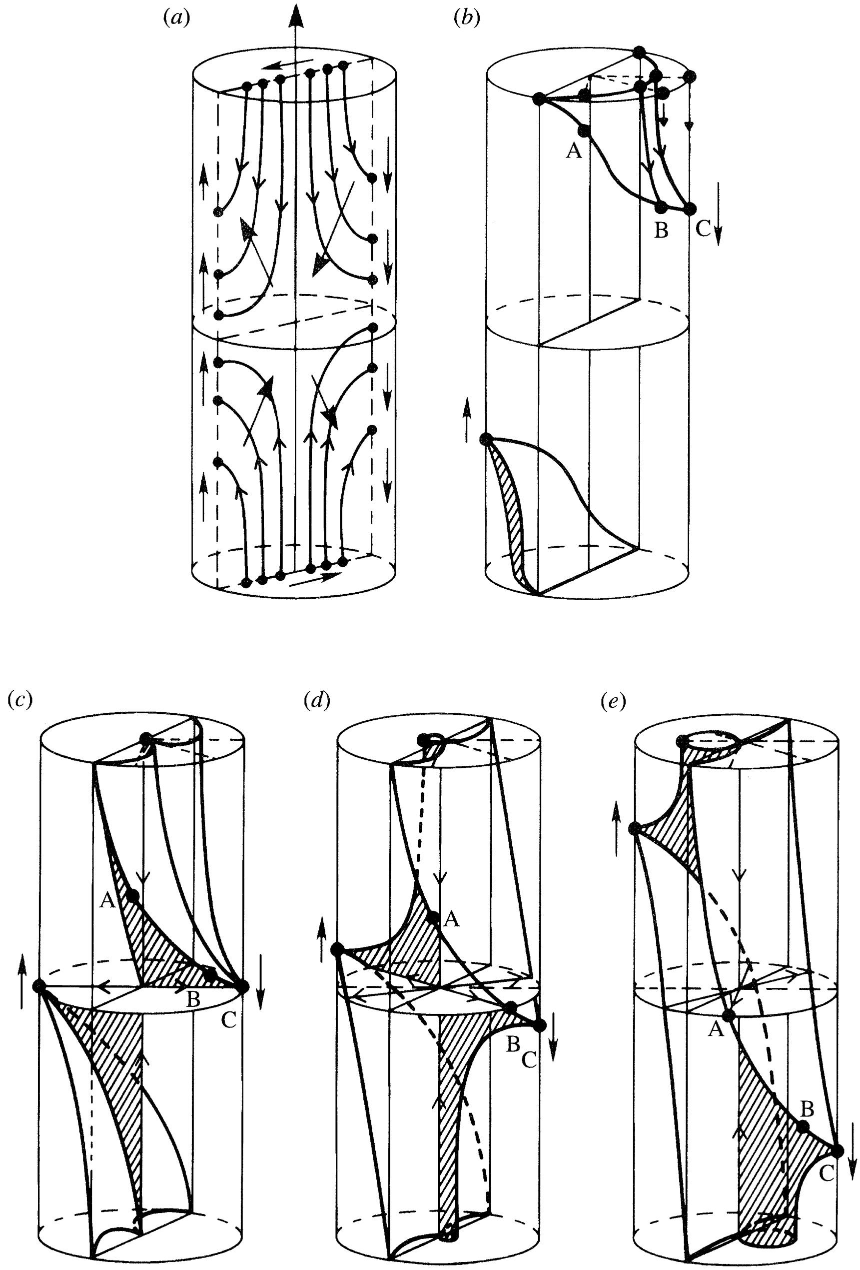

Figure 3. Spine reconnection showing (a) field line motion in a plane $\phi =$ const. and (b)–(e) the motion of flux surfaces.

Now consider the field lines in a plane $\phi =$ const. (figure 3a) and suppose their footpoints on a cylindrical surface $R = 1$ move vertically up or down: upwards for those on the left and downwards for those on the right. Then the field lines will simply approach the null-point, break and reconnect as the footpoints cross the plane $z = 0$, and then move away from the null point in the standard way that is familiar in two dimensions. The other ends of the field lines on the top surface $z = 1$ simply move from right to left through the z-axis, while those on the bottom surface move from left to right.

Next, suppose the same process is taking place in all the other planes $\phi = \text{constant}$ and consider the implications for flux surfaces that come into the cylindrical volume $|R| \leqslant 1$, $|z| \leqslant 1$ by first crossing the boundary of the volume in the curves $R = 1$, $z = \pm 1$. Clearly, since the field lines at one value of ϕ have the opposite radial velocity from those at $\phi + \pi$, the whole process is modulated in ϕ and there must be some surface (say $\phi = 0$ and $\phi = \pi$) on which the field lines are stationary. Thus, consider the flux surfaces in the volume $0 < \phi < 1$, $0 < R < 1$, $0 < z < 1$, which intersect the curved surface in a curve that moves downwards and the top surface in one that moves inwards (figure 3*b*). The flux surface moves inwards and distorts until it touches the origin and creates a fold along the spine (z-axis); see figure 3*c*. At the same time a symmetrically placed flux surface below $z = 0$ and to the left of $\phi = 0$ comes in and distorts until it too touches the origin and the lower half of the spine (in a fold). The surfaces are then cut and reconnect and a bubble expands from the fold on the other side of the spine (figure 3*d*). During this process the flow velocity in planes $z = \text{const.}$ is radial and converges on the spine (the z-axis) and so we expect singular behaviour there and refer to this process as 'spine reconnection'.

In order to model the above behaviour of the field (4.2), we suppose the velocity has only R- and z-components

$$\boldsymbol{v}(R, \phi, z) = v_R \hat{\boldsymbol{R}} + v_z \hat{\boldsymbol{z}}. \tag{4.3}$$

Then the electric field equation (2.1) implies that

$$\boldsymbol{E}(R, \phi, z) = E\hat{\boldsymbol{\phi}},$$

while the components of Faraday's law (2.2) are

$$\frac{\partial E}{\partial z} = 0, \qquad \frac{1}{R}\frac{\partial}{\partial R}(RE) = 0,$$

so that

$$E = \frac{E_0(\phi)}{R}. \tag{4.4}$$

Equation (2.3) may then be written

$$\frac{E_0}{R} = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi},$$

and so, once $E_0(\phi)$ is known, the electric potential may be deduced from

$$\Phi = -\int E_0 \, \mathrm{d}\phi. \tag{4.5}$$

For example, a reasonable form for $E_0(\phi)$ is

$$E_0(\phi) = v_{\mathrm{e}} \sin \phi,$$

where v_{e} is the field line speed on the cylinder $R = 1$.

Finally, the field line velocity ($\boldsymbol{v}_\perp$) from (2.5) has components

$$v_{\perp R} = \frac{2E_0(\phi) z}{R(R^2 + 4z^2)}, \qquad v_{\perp \phi} = 0, \qquad v_{\perp z} = \frac{E_0(\phi)}{R^2 + 4z^2}. \tag{4.6}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

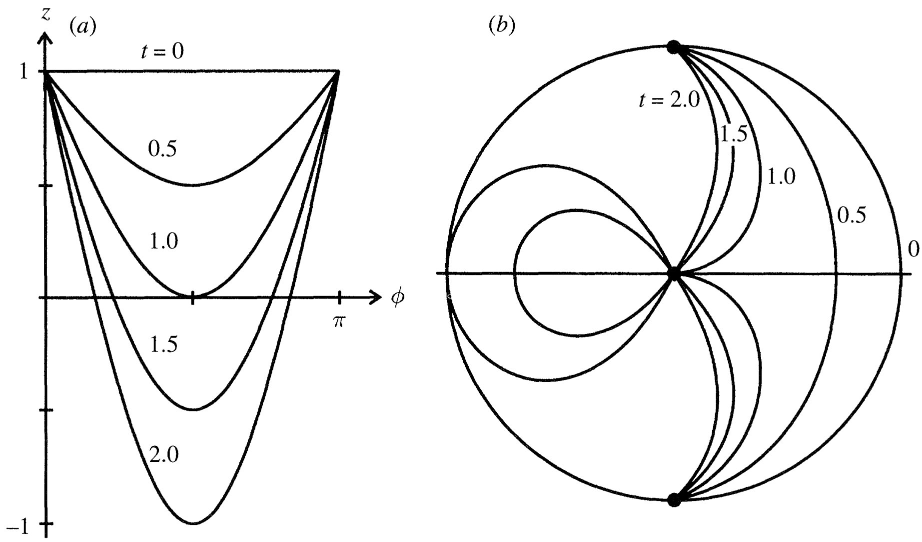

Figure 4. Trace of field line footpoints on (*a*) the curved surface $R = 1$ and (*b*) the top surface $z = 1$.

The magnitude of $\boldsymbol{v}_\perp$ is

$$v_\perp = \frac{E}{B} = \frac{E_0(\phi)}{R(R^2 + 4z^2)^{1/2}}. \tag{4.7}$$

Thus, we note that, as well as being singular at the null point where B vanishes, the radial component of $\boldsymbol{v}_\perp$ is singular all along the spine on the z-axis since the electric field becomes singular there.

From (4.2) the equations of the field lines are

$$\phi = \text{const.}, \quad R^2 z = \text{const.}$$

Flux surfaces are envelopes of these field lines in a similar way that a caustic is an envelope of straight lines. Suppose, in particular, that the footpoints meet the cylindrical surface $R = 1$ at a height

$$z = 1 - t \sin\phi.$$

Then such field lines move (down for $0 < \phi < \pi$) at speed $\dot{z} = -\sin\phi$ and trace out the curves shown in figure 4*a*. The flux surface through them is

$$R^2 z = 1 - t \sin\phi. \tag{4.8}$$

They meet the top surface $z = 1$ in the curve

$$R^2 = 1 - t \sin\phi \tag{4.9}$$

for $t < 1/\sin\phi$ and $0 < \phi < \pi$. The footpoints that start from the bottom surface $z = -1$ and move up between $\phi = \pi$ and $\phi = 2\pi$ are located on $R = 1$ at

$$z = -1 - t \sin\phi$$

and so trace out the surface

$$R^2 z = -1 - t \sin\phi,$$

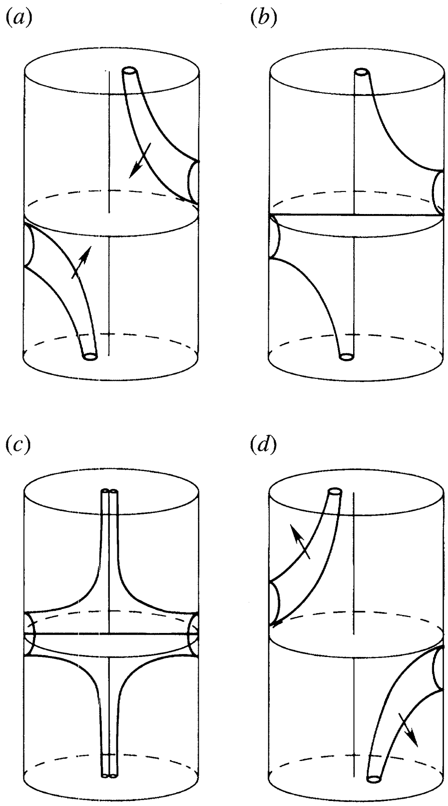

Figure 5. Spine reconnection of a curved flux tube.

which meets the top $z = 1$ in the curve

$$R^2 = -1 - t \sin\phi \tag{4.10}$$

when $-t \sin\phi > -1$ and $|\phi - 3\pi/2| < -1/\sin\phi$. The resulting curves on the top surface are shown in figure 4*b*.

Of course we may define a flux surface to be the surface formed from field lines that pass through any curve we like. So for example if we consider a closed curve of footpoints moving down the cylinder $R = 1$, then a curved flux tube will move in and reconnect as the footpoints cross the fan surface, as shown in figure 5.

So far we have calculated the flow velocity normal to the field lines, but, as the field lines move, the flow $v_{||}$ along the field changes. It may be calculated if an extra assumption is adopted such as incompressibility so that

$$\nabla \cdot \boldsymbol{v} = 0. \tag{4.11}$$

However, in this case we may proceed to calculate the total plasma velocity directly and hence by inference the parallel component. Thus, in view of (4.11), the velocity may be written in terms of a stream function (Ψ) as

$$v_R = -\frac{\partial \Psi}{\partial z}, \qquad v_z = \frac{1}{R}\frac{\partial}{\partial R}(R\Psi),$$

so that the streamlines are given by $\mathrm{d}z/\mathrm{d}r = v_z/v_R$ or

$$\tilde{\Psi} \equiv R\Psi = \text{const.} \tag{4.12}$$

Then the electric field equation (2.1) may be written

$$E + v_R B_z - v_z B_R = 0,$$

or

$$E_0 - \frac{\partial \tilde{\Psi}}{\partial z} B_z - \frac{\partial \tilde{\Psi}}{\partial R} B_R = 0. \tag{4.13}$$

Now the magnetic field lines are given by $\mathrm{d}z/\mathrm{d}R = B_z/B_R$ or

$$R^2 z = \text{const.}, \tag{4.14}$$

and along such field lines the solution of (4.13) is

$$\tilde{\Psi} = \int \frac{E_0\,\mathrm{d}s}{B} = E_0 \int \frac{\mathrm{d}R}{B_R}.$$

By putting $B_R = R$ and making use of (4.14) this may be integrated to give

$$\Psi = \frac{E_0}{R}(\log R + f(R^2 z)).$$

If in particular we want $R = z$ to be a stream line and put $\Psi = 0$ there, the function f is determined to be $f(u) = -\frac{1}{3}\log u$ and the stream function becomes

$$\Psi = \frac{E_0}{3R}\log\frac{R}{z}. \tag{4.15}$$

This in turn determines the velocity components to be

$$v_R = \frac{E_0}{3Rz}, \qquad v_z = \frac{E_0}{3R^2},$$

so that in planes $\phi = \text{const.}$ we have stream lines with purely inflow in the first and third quadrants (for $E_0 < 0$) and outflow in the second and fourth.

(b) *Fan reconnection*

In the previous section we have given an example of how, when continuous footpoint motions are imposed on a cylindrical surface (such as $R = 1$) surounding the spine, then a singular motion is driven at the spine axis. In contrast, if we impose continuous motions on surfaces (such as $z = \pm 1$) that cut across the spine, then singular behaviour is instead driven at the fan surface.

Suppose, for example, that the footpoints on the top surface ($z = 1$) move in a straight line from right to left (figure 6*a*). Then the other ends of the field lines in the curved surface $R = 1$ twirl around the z-axis like a swirling skirt. As the footpoint crosses the $\phi = 0$ plane on the top surface, the field line lies entirely in the plane $\phi = 0$.

Next consider the flux surface made of a set of such field lines whose footpoints march across the top surface in a straight line (figure 6*b*). As the straight line moves towards the spine axis, the flux surface distorts as its lower ends rotate from right to left. In the limit when the line of footpoints meet the z-axis, the flux surface becomes the vertical surface $\phi = 0$ above $z = 0$, together with the horizontal half-plane $z = 0$, $0 < \phi < \pi$. At the same time a symmetrically placed flux surface to the left of $\phi = 0$ and below $z = 0$ moves upward and becomes the vertical flux surface below $z = 0$ together with the half-plane $z = 0$, $\pi < \phi < 2\pi$. The surfaces then break from their

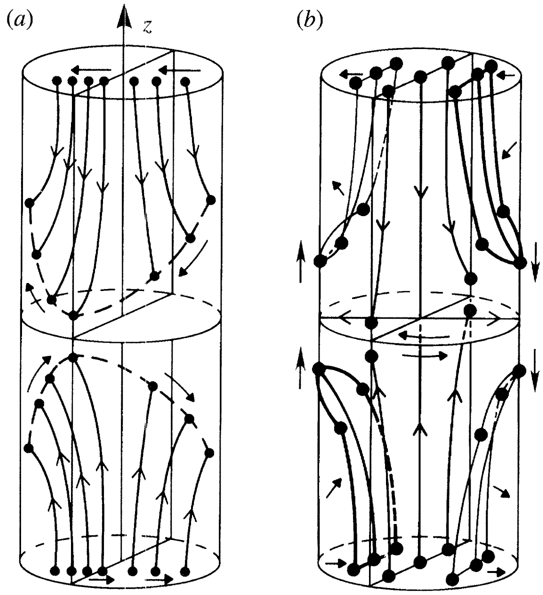

Figure 6. Fan reconnection showing (*a*) field line motion and (*b*) flux surface motion.

lines $\phi = \frac{1}{2}\pi$, $z = 0$, reconnect and move away as shown. During this process the field lines above the fan surface ($z = 0$) rotate rapidly in one direction while those below it rotate in the opposite direction. The fan therefore represents a singular surface, which is why we are referring to this process as 'fan reconnection'.

How do we model the above process mathematically? The magnetic field lines for our radial null field (4.1) are given by

$$\frac{\mathrm{d}x}{B_x} = \frac{\mathrm{d}y}{B_y} = \frac{\mathrm{d}z}{B_z},$$

or

$$y = \frac{y_0}{x_0}x, \quad z = \frac{x_0^2 z_0}{x^2}, \tag{4.16}$$

where x_0, y_0, z_0 are constants. In general we may impose the value of $\Phi_0 = \Phi(x_0, y_0, z_0)$ on a surface $z_0 = z_0(x_0, y_0)$, say, and then deduce $\Phi(x, y, z)$ as having the same value on each field line of the form (4.16). The values of $\boldsymbol{E} = -\nabla\phi$ and $\boldsymbol{v}_\perp = \boldsymbol{E} \times \boldsymbol{B}/B^2$ follow as usual. Thus a general question that arises is: what forms for Φ_0 or E can we impose and which ones produce reconnection? In the present case the problem is to determine what form of Φ_0, if any, makes $v_{\perp x} = 0$ and $v_{\perp y}$ independent of x on $z = 1$, so that the footpoints move across in a straight line. If we regard $(x_0, y_0, 1)$ as being the coordinates of such footpoints, then the equations of the field lines are

$$y = \frac{y_0}{x_0}x, \quad x^2 z = x_0^2. \tag{4.17}$$

Thus as the point $(x_0, y_0, 1)$ moves across the top surface with x_0 constant and y_0 decreasing, so the other end of the field line moves along the surface of the cylinder $x^2 + y^2 = 1$ with $z = x_0^2/x^2$.

If we then impose

$$\Phi_0 = f(x_0, y_0) \quad \text{on} \quad z = 1,$$

the relations (4.17) may be inverted to give

$$x_0 = x\sqrt{z}, \quad y_0 = y\sqrt{z},$$

and so the form of Φ throughout the volume threaded by these field lines is

$$\Phi = f(x\sqrt{z}, y\sqrt{z}). \tag{4.18}$$

The resulting electric field components are

$$E_x = -\sqrt{z}\frac{\partial f}{\partial x_0}, \quad E_y = -\sqrt{z}\frac{\partial f}{\partial y_0}, \quad E_z = -\frac{x}{2\sqrt{z}}\frac{\partial f}{\partial x_0} - \frac{y}{2\sqrt{z}}\frac{\partial f}{\partial y_0},$$

and so the condition that

$$v_{\perp x} = \frac{E_y B_z - E_z B_y}{B^2}$$

vanish on $z = 1$ becomes

$$(4 + y_0^2)\frac{\partial f}{\partial y_0} + x_0 y_0 \frac{\partial f}{\partial x_0} = 0. \tag{4.19}$$

This equation has a general solution

$$f = f\left(\frac{4 + y_0^2}{x_0^2}\right),$$

and so the solution (4.18) becomes

$$\Phi = f\left(\frac{4 + y^2 z}{x^2 z}\right). \tag{4.20}$$

Then the component

$$v_{\perp y} = \frac{E_z B_x - E_x B_z}{B^2}$$

becomes on $z = 1$

$$v_{\perp y} = -\frac{4}{x_0^3} f'\left(\frac{4 + y_0^2}{x_0^2}\right).$$

For this to be independent of x_0 we put

$$f(u) = \frac{c}{u^{1/2}} \tag{4.21}$$

and so

$$v_{\perp y} = \frac{2c}{(4 + y^2)^{3/2}} \tag{4.22}$$

on $z = 1$.

Thus our final solution for the potential is

$$\Phi = \left(\frac{x^2 z}{4 + y^2 z}\right)^{1/2}, \tag{4.23}$$

and the corresponding electric field components are

$$E_x = \frac{z^{1/2}}{(4 + y^2 z)^{1/2}}, \quad E_y = -\frac{xyz^{3/2}}{(4 + y^2 z)^{3/2}}, \quad E_z = \frac{2xz^{-1/2}}{(4 + y^2 z)^{3/2}}. \tag{4.24}$$

The resulting velocity components are

$$\left.\begin{aligned} v_{\perp x} &= \frac{2xy(z^3-1)}{(x^2+y^2+4z^2)(4+y^2z)^{3/2}z^{1/2}}, \\ v_{\perp y} &= \frac{2(x^2+4z^2+y^2z^3)}{(x^2+y^2+4z^2)(4+y^2z)^{3/2}z^{1/2}}, \\ v_{\perp z} &= \frac{(4+y^2z+x^2z)yz^{1/2}}{(x^2+y^2+4z^2)(4+y^2z)^{3/2}}. \end{aligned}\right\} \tag{4.25}$$

From this solution we can check that, as required, $v_{\perp x} = 0$ and $v_{\perp y}$ is a non-singular function of y alone on the top surface $z = 1$, namely (4.22). As the fan ($z = 0$) is approached, $v_{\perp z}$ tends to zero, but $v_{\perp x}$ and $v_{\perp y}$ tend to infinity like $z^{-1/2}$. On the z-axis $\boldsymbol{v}_\perp = 1/(4z^{1/2})\hat{\boldsymbol{y}}$, while on the cylinder $x^2 + y^2 = 1$

$$\boldsymbol{v}_\perp = \frac{1}{(1+4z^2)(4+y^2z)^{3/2}} \times \left(\frac{2y}{z^{1/2}}(z^3-1)\sqrt{(1-y^2)}, \frac{2}{z^{1/2}}(1-y^2+4z^2+y^2z^3), (4+z)yz^{1/2} \right). \tag{4.26}$$

5. Resolution of singularities

(a) Anti-reconnection theorem

Having described physically the process of reconnection near a null point and determined the motion in the ideal limit, the next stage is to try and resolve the singularities that occur at the null and along the spine and fan by the inclusion of diffusion processes. While such a complete resolution is outside the scope of this paper, we present here some powerful constraints and make some qualitative comments. The natural procedure is simply to focus first of all on linear reconnection (Priest *et al.* 1994) in which one assumes a slow flow everywhere and linearises about the potential null-point field by writing $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1$, $\boldsymbol{E} = \boldsymbol{E}_1$, $\boldsymbol{j} = \boldsymbol{j}_1$, $\boldsymbol{v} = \boldsymbol{v}_1$ (where we drop the subscripts on $\boldsymbol{E}$, $\boldsymbol{j}$, $\boldsymbol{v}$ in the following proof). In the process of seeking such solutions we came across great difficulty and discovered an 'antireconnection' theorem which limits greatly the type of solutions that we are allowed to seek.

Theorem 5.1. *Steady MHD reconnection in three dimensions with convective plasma flow across the spine or fan of a proper radial null point is impossible in an inviscid plasma with a highly sub-Alfvenic flow and a uniform magnetic diffusivity.*

Proof. The basic equations are as follows. For resistive MHD flow Ohm's law is

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}_0 = \eta \boldsymbol{j}, \tag{5.1}$$

where

$$\boldsymbol{j} = \nabla \times \boldsymbol{B}_1/\mu.$$

Equation (5.1) implies that

$$\boldsymbol{B}_0 \cdot \boldsymbol{E} = \eta \boldsymbol{B}_0 \cdot \boldsymbol{j}, \tag{5.2}$$

and the plasma flow normal to the magnetic field is

$$\boldsymbol{v}_\perp = \frac{(\boldsymbol{E} - \eta \boldsymbol{j}) \times \boldsymbol{B}_0}{B_0^2},$$

which for our proper radial null point with field

$$(B_{0R}, B_{0\phi}, B_{0z}) = (R, 0, -2z) \tag{5.3}$$

becomes

$$\boldsymbol{v}_{\perp} = \frac{(\eta j_{\phi} - E_{\phi})}{B_0^2}(2z, 0, R). \tag{5.4}$$

For an inviscid plasma whose flow speed is everywhere much slower than the Alfven speed, the equation of motion reduces to the force balance

$$\boldsymbol{j} \times \boldsymbol{B}_0 = \nabla p,$$

whose curl is

$$\nabla \times (\boldsymbol{j} \times \boldsymbol{B}_0) = \mathbf{0}.$$

This may be expanded out (using $\nabla \cdot \boldsymbol{j} = \nabla \cdot \boldsymbol{B}_1 = 0$) to give

$$(\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{j} = (\boldsymbol{j} \cdot \nabla)\boldsymbol{B}_0. \tag{5.5}$$

The final equation is Faraday's law for a steady process

$$\nabla \times \boldsymbol{E} = 0. \tag{5.6}$$

Thus for linear reconnection at a null point with field (5.3), equations (5.4), (5.5), (5.6) and (5.2) determine respectively the flow $\boldsymbol{v}_{\perp}$, electric current $\boldsymbol{j}$ and electric field $\boldsymbol{E}$.

Consider these equations first at the fan plane $z = 0$, where the flow (5.4) becomes

$$v_{\perp z} = \frac{1}{R}(\eta j_{\phi} - E_{\phi}), \tag{5.7}$$

and so we need to find j_{ϕ} and E_{ϕ}. The ϕ-component of (5.5) becomes

$$R\frac{\partial j_{\phi}}{\partial R} = j_{\phi},$$

which determines the azimuthal current as

$$j_{\phi} = f(\phi)R.$$

The z-component of (5.6) relating E_{ϕ} and E_R becomes

$$R\frac{\partial E_{\phi}}{\partial R} + E_{\phi} = \frac{\partial E_R}{\partial \phi}, \tag{5.8}$$

where (5.2) is

$$E_R = \eta j_R,$$

and the R-component of (5.5), namely

$$R\frac{\partial j_R}{\partial R} = j_R,$$

determines j_R as

$$j_R = g(\phi)R.$$

Thus (5.8) may be solved to give

$$E_{\phi} = \tfrac{1}{2}\eta g'(\phi)R + \frac{h(\phi)}{R},$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and so (5.7) becomes

$$v_{\perp z} = \eta[f(\phi) - \tfrac{1}{2}g'(\phi)] - h(\phi)/R^2. \tag{5.9}$$

For regular and continuous behaviour at $R = 0$ we need, respectively, $h(\phi) \equiv 0$ and $f(\phi) - \frac{1}{2}g'(\phi) = \text{const}$. So $v_{\perp z}$ is simply constant at the fan plane; it has a value proportional to resistivity η and is therefore diffusive since it vanishes in the limit as η tends to zero. Thus convective flow across the fan (which would not vanish in such a limit) is not allowed, and so spine reconnection cannot be driven by such a flow: it can only be caused by magnetic diffusion across the fan. In the next section we shall consider an explicit example of such a flow.

The second part of the proof – to consider the behaviour at the spine $R = 0$ – is most easily carried out in a rectangular system of coordinates, where the flow has a y-component of velocity

$$v_{\perp y} = (E_x - \eta j_x)/(2z) \tag{5.10}$$

and so we need to find E_x and j_x at the spine. The x-component of (5.5) becomes

$$-2z\frac{\partial j_x}{\partial z} = j_x,$$

which determines j_x as

$$j_x = \frac{c_1}{|z|^{1/2}}. \tag{5.11}$$

In turn, E_x is determined in general by eliminating E_z between (5.2) and the y-component of (5.6), namely,

$$xE_x + yE_y - 2zE_z = \eta(xj_x + yj_y - 2zj_z)$$

and

$$\frac{\partial E_z}{\partial x} = \frac{\partial E_x}{\partial z},$$

to give on $x = y = 0$

$$2z\frac{\partial E_x}{\partial z} - E_x = \eta\left(2z\frac{\partial j_z}{\partial x} - j_x\right). \tag{5.12}$$

In this equation j_z is given from the z-component of (5.5), namely

$$x\frac{\partial j_z}{\partial x} + y\frac{\partial j_z}{\partial y} - 2z\frac{\partial j_z}{\partial z} = -2j_z$$

as

$$j_z = zH(\xi, \zeta), \tag{5.13}$$

where $H(\xi, \zeta)$ is an arbitrary analytical function of $\xi = y^2 z$ and $\zeta = z(x^2 + y^2)$, so that

$$\frac{\partial j_z}{\partial x} = 2xz\frac{\partial H}{\partial \zeta}$$

vanishes at $x = y = 0$. Thus, on account of (5.11), we obtain from (5.12) and (5.10)

$$E_x = \frac{\eta c_1}{2|z|^{1/2}} + c_2|z|^{1/2}, \tag{5.14}$$

$$v_{\perp y} = \left(-\frac{\eta c_1}{4|z|^{3/2}} + \frac{c_2}{2|z|^{1/2}}\right)\text{sgn}(z). \tag{5.15}$$

For regular behaviour of $v_{\perp y}$ at $z = 0$ we need $c_1 = c_2 = 0$, and so $v_{\perp y}$ vanishes at the spine. In other words, there is no flow across the spine line and fan reconnection is impossible when nonlinear and viscous effects are negligible. ∎

(b) *Some qualitative features of a nonlinear model*

In view of the antireconnection theorem it is not possible to set up the simplest linear model of three-dimensional reconnection. It is therefore likely that nonlinear effects will be needed to resolve the singularities at the spine, the null itself and the fan. Although such a resolution is outside the scope of the present paper and is a matter for future research, it is worth making a few qualitative comments.

First of all, consider the spine singularity, which for $z \gg R$ has an electric field and field line velocity of

$$E_\phi \approx \frac{v_e \sin\phi}{R}, \quad v_{\perp R} \approx \frac{v_e \sin\phi}{2zR}. \tag{5.16}$$

Inertia becomes important at a radius where the flow speed becomes in order of magnitude as large as the Alfven speed $v_A = 2zv_{AE}$, where v_{AE} is the Alfven speed at $z = 0$ on the cylinder $R = 1$, i.e. at

$$R = \frac{v_e}{v_{AE}} \frac{\sin\phi}{4z^2}.$$

This represents two cylindrical surfaces touching at the z-axis and having cross sections that are decreasing as z^{-2}. Viscosity would become important before inertia only if the Reynolds number were less than unity; also, if resistive diffusion in the ϕ-component of Ohm's law were $\eta j_\phi \approx \eta \partial B_z/\partial R \approx \eta B_e/R$, then it would never become of order $E_\phi \approx v_e/R$ if diffusion were not important at $R = 1$.

Inertia may perhaps in principle modify $v_{\perp R}$ and introduce an azimuthal component $v_{\perp\phi}$ with its associated E_R. Although the actual forms would in principle be determined by the equations of motion and induction, the qualitative effect can be seen by choosing

$$E_R = -\frac{l\cos\phi}{(l+R)^2}, \tag{5.17}$$

so that from the z-component of (5.6)

$$E_\phi = \frac{\sin\phi}{l+R},$$

where l is the width of the singularity and l^{-1} the maximum electric field. The resulting flow velocity from (2.5) has components

$$(v_{\perp R}, v_{\perp\phi}, v_{\perp z}) = -\frac{1}{B^2}\left(\frac{2z\sin\phi}{l+R}, \frac{2zl\cos\phi}{(l+R)^2}, \frac{R\sin\phi}{l+R}\right). \tag{5.18}$$

Secondly, consider the vicinity of the null point. Since $\boldsymbol{B}$ vanishes at the null itself, the existence of a non-zero electric field implies from (5.1) that the current does not vanish and so locally the field will be distorted from the form (5.3) we have been considering throughout this work. Now Taylor expand the field, so that to lowest order

$$(B_R, B_\phi, B_z) = (aR + bz, cR + dz, eR + fz), \tag{5.19}$$

where a, b, c, d, e, f are functions of ϕ.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Assume further that B_z vanishes at $z = 0$ so that $e = 0$. Then the equation $\nabla \cdot \boldsymbol{B} = 0$ becomes

$$2a + \frac{bz}{R} + f + c' + \frac{d'z}{R} = 0,$$

which implies that

$$2a + f + c' = 0, \quad b + d' = 0.$$

Furthermore, the components of the electric current are

$$(j_R, j_\phi, j_z) = (f'z/R - d, b, 2c - a' + (d - b')z/R), \tag{5.20}$$

so that, in order to have a finite current on $R = 0$, we need

$$f' = 0, \quad d = b'.$$

For simplicity, let us assume that $a = 1$ and $c' = 0$; then the field and current become with $b = b_0 \sin\phi$

$$(B_R, B_\phi, B_z) = (R + b_0 z \sin\phi, \quad c_0 R + b_0 z \cos\phi, \quad -2z),$$
$$(j_R, j_\phi, j_z) = (-b_0 \cos\phi, \quad b_0 \sin\phi, 2c_0).$$

Now, as we have seen, spine reconnection is associated with an E_ϕ and so if we suppose this produces a j_ϕ near the null the simplest solution is to set $c_0 = 0$, so that the field and current just become

$$(B_x, B_y, B_z) = (x, y + b_0 z, -2z), \qquad (j_x, j_y, j_z) = (-b_0, 0, 0). \tag{5.21}$$

In other words the current is locally uniform in the y-direction and the field lines are distorted in the yz-plane so that the spine bends away from the normal in the y-direction.

In contrast, fan reconnection is associated with an E_z and would tend to create a j_z-component at the null. The simplest solution here is to set $b_0 = 0$ so that

$$(B_R, B_\phi, B_z) = (R, c_0 R, -2z), \qquad (j_R, j_\phi, j_z) = (0, 0, 2c_0), \tag{5.22}$$

and the field lines near the null spiral about the z-axis.

Finally, consider briefly the singularity at the fan where the azimuthal velocity behaves according to (4.26) like

$$v_\phi = \frac{\cos\phi}{4z^{1/2}}$$

near the fan surface. It may be possible for such a singularity to be smoothed out by viscosity, with the Lorentz force associated with a twist in the field lines slowing down the plasma. Thus, if one considers azimuthal flow and field components $v_\phi(z)$ and $B_\phi(z)$, the resulting ϕ-component of a force-balance between viscous and magnetic forces near the fan plane $(\partial/\partial z \gg R^{-1})$ is

$$\nu v_\phi'' = 2zB_\phi' - B_\phi.$$

A linear variation of $B_\phi = cz$ (associated with field lines twisting one way above the plane and the opposite way below the plane) would therefore give a velocity profile of the form

$$v_\phi = \frac{c}{\nu}(\tfrac{1}{3}z^3 - dz)$$

near the plane, as required. However, any further discussion of the effect of viscosity is beyond the scope of this paper and has only just been understood in two dimensions (Priest *et al.* 1994; Titov & Priest 1996).

(c) *Smoothing effects of resistivity*

It is clear from the previous considerations that the movement of the magnetic footpoints on the boundary of the volume containing a null point may lead to the appearance of a singularity in the plasma velocity near the null if the magnetic field is frozen to the plasma. It is natural to hope that finite resistivity could resolve such a singularity. In two dimensions such a hope is indeed fulfilled for non-steady flow (Craig & McClymont 1991), but for steady linear flow the extra effects of viscosity or nonlinearity are required (Craig & Rickard 1994; Priest *et al.* 1994; Titov & Priest 1996), although in the case of nonlinearity there is so far only the suggestive results of numerical experiments (Biskamp 1986; Priest & Forbes 1992) rather than a formal proof. Here we are investigating whether resistivity alone can resolve the singularity in the steady, linear, three-dimensional case and find that for spine reconnection we are still left with a discontinuity in v_z at the spine.

To investigate the problem in its simplest form, suppose that the flow is slow enough to consider it as a perturbation of an initially potential configuration (4.1) so that inertia is negligible. Assume viscosity is also negligible, so that the perturbed equation of motion reduces to the magnetostatic force balance

$$-\nabla p + \boldsymbol{j} \times \boldsymbol{B}_0 = 0, \tag{5.23}$$

where $\boldsymbol{B}_0$ is the unperturbed potential magnetic field with components (4.1) or (4.2). The perturbation of the magnetic field $\boldsymbol{B}$ associated with the current $\boldsymbol{j}$ is assumed to be small due to the smallness of $\boldsymbol{j}$ which, in turn, is due to the slowness of the flow. Since $\boldsymbol{j} = \nabla \times \boldsymbol{B}/\mu$ we should require also that

$$\nabla \cdot \boldsymbol{j} = 0, \tag{5.24}$$

which implies that there is no accumulation of charge in the volume.

For finite resistivity, instead of (2.1)–(2.3), we have

$$-\nabla \Phi + \boldsymbol{v} \times \boldsymbol{B}_0 = \eta \boldsymbol{j}, \tag{5.25}$$

where $\eta \ll 1$ is a dimensionless resistivity or inverse magnetic Reynolds number. We can restrict our consideration also to incompressible flows by putting

$$\nabla \cdot \boldsymbol{v} = 0. \tag{5.26}$$

Thus equations (5.23)–(5.26) form the basic system describing slow steady incompressible resistive MHD flows in the neighbourhood of a three-dimensional magnetic null point.

(i) *General solution of the basic equations*

One can find the general solution for the current density $\boldsymbol{j}$ and the pressure p from (5.23) and (5.24) as follows. Equation (5.23) yields an expression for the current density $\boldsymbol{j}_\perp$ perpendicular to the magnetic field $\boldsymbol{B}_0$

$$\boldsymbol{j}_\perp = \frac{\boldsymbol{B}_0 \times \nabla p}{B_0^2}, \tag{5.27}$$

where the pressure p must satisfy

$$\boldsymbol{B}_0 \cdot \nabla p = 0, \tag{5.28}$$

which is obtained from the scalar product of $\boldsymbol{B}_0$ with (5.23). This equation implies that the pressure is constant along the unperturbed magnetic field lines and physically it corresponds to the case when the plasma moves so slowly that the pressure perturbations are smoothed out along the field lines. According to (4.2) in a cylindrical system of coordinates it is written as

$$R\frac{\partial p}{\partial R} - 2z\frac{\partial p}{\partial z} = 0 \tag{5.29}$$

and may be easily solved by the method of characteristics to give

$$p = p(\phi, \zeta), \qquad \zeta = R^2 z. \tag{5.30}$$

Here ζ is constant along an unperturbed magnetic field line and so plays the role of labelling different field lines in planes $\phi = \text{const}$.

Equation (5.24) then determines the longitudinal component of the current density

$$\boldsymbol{j}_{\|} \equiv j_{\|}\frac{\boldsymbol{B}_0}{B_0}, \tag{5.31}$$

since it can be written as

$$\nabla \cdot \boldsymbol{j}_{\|} = -\nabla \cdot \boldsymbol{j}_{\perp}. \tag{5.32}$$

By using (5.27)–(5.30) the right-hand side of this equation becomes

$$\begin{aligned} -\nabla \cdot \boldsymbol{j}_{\perp} &= \nabla \cdot \left(\frac{\nabla p \times \boldsymbol{B}_0}{B_0^2}\right) \\ &= \frac{12z}{B_0^4}\frac{\partial p}{\partial \phi}. \end{aligned}$$

Here we have also used the facts that $\nabla \times \nabla p = 0$ and $\nabla \times \boldsymbol{B}_0 = 0$. Due to (5.31) and $\nabla \cdot \boldsymbol{B}_0 = 0$, the left-hand side of (5.32) reduces to

$$\nabla \cdot \left(\frac{j_{\|}}{B_0}\boldsymbol{B}_0\right) = \boldsymbol{B}_0 \cdot \nabla \left(\frac{j_{\|}}{B_0}\right), \tag{5.33}$$

so that the resulting equation for $j_{\|}$ is

$$\boldsymbol{B}_0 \cdot \nabla \left(\frac{j_{\|}}{B_0}\right) = \frac{12z}{B_0^4}\frac{\partial p}{\partial \phi}. \tag{5.34}$$

To solve this equation we can again apply the method of characteristics: in the system of coordinates (R, ϕ, ζ) it may be written as

$$R\frac{\partial}{\partial R}\left(\frac{j_{\|}}{B_0}\right) = \frac{12R^6\zeta}{(R^6 + 4\zeta^2)^2}\frac{\partial p}{\partial \phi}(\phi, \zeta),$$

so that after integration we have

$$j_{\|} = \alpha B_0 - \frac{2z}{R^2 B_0}\frac{\partial p}{\partial \phi}. \tag{5.35}$$

The first term here represents the force-free component of the current density with an arbitrary function $(\alpha(\phi, \zeta))$ that is constant along magnetic field lines. The second term determines that part of the longitudinal current which has a divergence: therefore, by equation (5.32) it is associated with a transverse current $(\boldsymbol{j}_{\perp})$ which in turn provides a magnetostaticforce balance for the configuration. Thus, the expressions (5.27), (5.30), (5.31) and (5.35) represent the general solution of equations (5.23) and (5.24).

Equations (5.25) and (5.26) for the velocity $\boldsymbol{v}$ and electric potential Φ can be solved in a similar way. From (5.25) we obtain an expression for the perpendicular component of velocity

$$\boldsymbol{v}_\perp = \frac{1}{B_0^2}\boldsymbol{B}_0 \times (\nabla\Phi + \eta\boldsymbol{j}_\perp)$$

and an equation for Φ

$$(\boldsymbol{B}_0\cdot\nabla)\Phi = -\eta B_0 j_\parallel;$$

this shows that the electric potential changes along the magnetic field lines owing to the resistive dissipation of the longitudinal current. Substituting (5.26) and (5.35) into these relations and making transformations we obtain

$$\boldsymbol{v}_\perp = \frac{\boldsymbol{B}_0\times\nabla\Phi}{B_0^2} - \frac{\eta}{B_0^2}\nabla p \tag{5.36}$$

and

$$(\boldsymbol{B}_0\cdot\nabla)\Phi = \eta\left(-\alpha B_0^2 + \frac{2z}{R^2}\frac{\partial p}{\partial\phi}\right). \tag{5.37}$$

In coordinates (R,ϕ,ζ) this equation has the form

$$R\frac{\partial\Phi}{\partial R} = \eta\left[-\alpha(\phi,\zeta)\left(R^2 + 4\frac{\zeta^2}{R^4}\right) + \frac{2\zeta}{R^4}\frac{\partial p}{\partial\phi}(\phi,\zeta)\right],$$

which is easily integrated to give

$$\Phi = \eta\left[(z^2 - \tfrac{1}{2}R^2)\alpha - \frac{1}{2}\frac{z}{R^2}\frac{\partial p}{\partial\phi}\right] + \Phi_0. \tag{5.38}$$

Here $\Phi_0 = \Phi_0(\phi,\zeta)$ is an arbitrary function representing the electric potential for the limiting case $(\eta = 0)$ of a perfectly conducting plasma; its value remains constant along the magnetic field lines defined by $\phi = \text{const.}_1$ and $\zeta = \text{const.}_2$. Thus changes in the electric potential Φ along field lines are only due to the first term in (5.38), which represents a voltage drop caused by the force-free and 'magnetostatic' components of the longitudinal current.

The longitudinal component of velocity

$$\boldsymbol{v}_\parallel = v_\parallel\frac{\boldsymbol{B}_0}{B_0} \tag{5.39}$$

is then determined from (5.26) in the same way as $j_\parallel$ was found from (5.24): first we rewrite

$$\nabla\cdot\boldsymbol{v}_\parallel = -\nabla\cdot\boldsymbol{v}_\perp$$

with the help of (5.36) and (5.39) in the form

$$\boldsymbol{B}_0\cdot\nabla\left(\frac{v_\parallel}{B_0}\right) = \frac{12z}{B_0^4}\frac{\partial\Phi}{\partial\phi} + \eta\left[-\frac{\nabla(B_0^2)\cdot\nabla p}{B_0^4} + \frac{1}{B_0^2}\nabla^2 p\right],$$

and then, after substituting (5.38) and $B_0^2 = R^2 + 4\zeta^2/R^4$ into this equation, we obtain in (R,ϕ,ζ)-coordinates

$$R\frac{\partial}{\partial R}\left(\frac{v_\parallel}{B_0}\right) = \frac{12\zeta}{R^2B_0^4}\frac{\partial\Phi_0}{\partial\phi} + \eta\left[R^2\frac{\partial^2 p}{\partial\zeta^2} + \frac{2}{R^6B_0^4}(2\zeta^2 - R^6)\left(3\zeta\frac{\partial\alpha}{\partial\phi} + 4\zeta\frac{\partial p}{\partial\zeta} - \frac{\partial^2 p}{\partial\phi^2}\right)\right].$$

Integration of this equation yields

$$v_{\|} = aB_0 - \frac{2z}{R^2B_0}\frac{\partial \Phi_0}{\partial \phi} + \eta\left[\frac{z}{2B_0}\left(3\frac{\partial \alpha}{\partial \phi} + 4\frac{\partial p}{\partial \zeta}\right) - \frac{1}{4R^2B_0}\frac{\partial^2 p}{\partial \phi^2} + \frac{1}{2}R^2B_0\frac{\partial^2 p}{\partial \zeta^2}\right]. \tag{5.40}$$

The first term (with an arbitrary function $a(\phi,\zeta)$) represents a flow that is constant along field lines and can be present even in the absence of transverse motion and currents. The second and third terms describe the longitudinal flow driven by the transverse convective and diffusive flows.

Thus the expressions for the velocity and potential are given by (5.36) and (5.38)–(5.40). They contain four arbitrary functions $p(\phi,\zeta)$, $\alpha(\phi,\zeta)$, $\Phi_0(\phi,\zeta)$ and $a(\phi,\zeta)$ which can be specified for a particular problem by appropriate boundary conditions.

Consider, for example, the problem in which the boundary values are prescribed at the cylindrical surface $R = 1$. For deducing the general solution in this case it is useful to write the components of $\boldsymbol{j}$ and $\boldsymbol{v}$ in cylindrical polars, for which the corresponding expressions become

$$j_R = R\alpha, \tag{5.41}$$

$$j_\phi = -R\frac{\partial p}{\partial \zeta}, \tag{5.42}$$

$$j_z = \frac{1}{R^2}\frac{\partial p}{\partial \phi} - 2z\alpha, \tag{5.43}$$

$$v_R = Ra + \frac{\eta}{4}\left(2\frac{z}{R}\frac{\partial \alpha}{\partial \phi} + 2R^3\frac{\partial^2 p}{\partial \zeta^2} - \frac{1}{R^3}\frac{\partial^2 p}{\partial \phi^2}\right), \tag{5.44}$$

$$v_\phi = -R\frac{\partial \Phi_0}{\partial \zeta} + \frac{\eta}{2}\left[R(R^2 - 2z^2)\frac{\partial \alpha}{\partial \zeta} + \frac{z}{R}\frac{\partial^2 p}{\partial \phi \partial \zeta} - \frac{1}{R^3}\frac{\partial p}{\partial \phi}\right], \tag{5.45}$$

$$v_z = \frac{1}{R^2}\frac{\partial \Phi_0}{\partial \phi} - 2za - \eta\left(\frac{1}{2}\frac{\partial \alpha}{\partial \phi} + \frac{\partial p}{\partial \zeta} + R^2 z\frac{\partial^2 p}{\partial \zeta^2}\right). \tag{5.46}$$

Since there are only four free functions in these expressions for six variables, we may generally fix just four of them at the boundary. It is natural to prescribe at the boundary the values of the velocity. At the cylindrical boundary surface one just needs to put $R = 1$ in (5.44)–(5.46) and take into account that $z = \zeta$ at this surface. The resulting equations

$$a + \frac{\eta}{4}\left(2\zeta\frac{\partial \alpha}{\partial \phi} + 2\frac{\partial^2 p}{\partial \zeta^2} - \frac{\partial^2 p}{\partial \phi^2}\right) = v_R|_{R=1} \equiv v_{bR}, \tag{5.47}$$

$$-\frac{\partial \Phi_0}{\partial \zeta} + \frac{\eta}{2}\left[(1 - 2\zeta^2)\frac{\partial \alpha}{\partial \zeta} + \zeta\frac{\partial^2 p}{\partial \phi \partial \zeta} - \frac{\partial p}{\partial \phi}\right] = v_\phi|_{R=1} \equiv v_b\phi, \tag{5.48}$$

$$\frac{\partial \Phi_0}{\partial \phi} - 2\zeta a - \eta\left(\frac{1}{2}\frac{\partial \alpha}{\partial \phi} + \frac{\partial p}{\partial \zeta} + \zeta\frac{\partial^2 p}{\partial \zeta^2}\right) = v_z|_{R=1} \equiv v_{bz} \tag{5.49}$$

determine three of the four initially arbitrary functions ($p(\phi,\zeta)$, $\alpha(\phi,\zeta)$, $\Phi_0(\phi,\zeta)$ and $a(\phi,\zeta)$) in terms of the components of velocity ($v_{bR}(\phi,\zeta)$, $v_{b\phi}(\phi,\zeta)$ and $v_{bz}(\phi,\zeta)$) given at the boundary $R = 1$. In principle, we could impose one more relation on the boundary values of current density, say, thereby fixing all the functional parameters in our solution. However, it is preferable to keep one of them free, so that we have

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

the freedom to try and resolve the above-mentioned singularity in the ideal plasma flows.

(ii) *Spine reconnection: partial resolution of the singularity for* $m = 1$ *and particular boundary conditions*

Consider now how the singularity appearing in the ideal spine reconnection process may be resolved by resistivity and a pressure perturbation. To make the problem as simple as possible, suppose that $v_{bR} \equiv 0$ and $v_{b\phi} \equiv 0$ in (5.47) and (5.48), so that in the limit $\eta \to 0$ we have

$$a = 0, \tag{5.50}$$

$$-\frac{\partial \Phi_0}{\partial \zeta} = 0, \tag{5.51}$$

$$\frac{\partial \Phi_0}{\partial \phi} - 2\zeta a = v_{bz}. \tag{5.52}$$

From here it follows that for ideal flows we may impose only $\Phi_0 \equiv \Phi_0(\phi)$ with $v_{bz} \equiv v_{bz}(\phi)$. It turns out that for resistive flows there is much more freedom in choosing v_{bz} (see the next section), but, for simplicity, we here assume $v_{bz} = v_{bz}(\phi)$ and in particular we consider the first harmonic,

$$v_{bz} = W \sin\phi, \tag{5.53}$$

of a Fourier expansion, where W = const. This condition implies that $v_{bz} \neq 0$ at $z = 0$ and therefore, according to our 'antireconnection' theorem, there must be no such solutions which are analytical at $R = 0$. However, it is interesting to see what happens to the flow near $R = 0$ if it has a non-vanishing transverse velocity at the fan plane.

Having adopted (5.53), the unknown functions may be assumed to have the following forms due to linearity of the problem:

$$p(\phi, \zeta) = P(\zeta)\sin\phi, \tag{5.54}$$

$$\alpha(\phi, \zeta) = A_1(\zeta)\cos\phi, \tag{5.55}$$

$$\Phi_0(\phi, \zeta) = F(\zeta)\cos\phi, \tag{5.56}$$

$$a(\phi, \zeta) = A_2(\zeta)\sin\phi. \tag{5.57}$$

After substituting these expressions into (5.47)–(5.49) we obtain

$$A_2 + \tfrac{1}{4}\eta(-2\zeta A_1 + 2P'' + P) = 0, \tag{5.58}$$

$$-F' + \tfrac{1}{2}\eta[(1 - 2\zeta^2)A_1' + \zeta P' - P] = 0, \tag{5.59}$$

$$-F - 2\zeta A_2 + \eta(\tfrac{1}{2}A_1 - P' - \zeta P'') = W, \tag{5.60}$$

where the primes denote differentiation with respect to ζ.

Let us now try to express the values A_1, A_2 and F in terms of the pressure amplitude P. Equation (5.58) yields

$$A_2 = \tfrac{1}{4}(2\zeta A_1 - 2P'' - P), \tag{5.61}$$

so that we can substitute this expression into (5.60) to give

$$F = -W + \tfrac{1}{2}\eta[(1 - 2\zeta^2)A_1 + (\zeta P - 2P')]. \tag{5.62}$$

Using this formula, we can derive from (5.59)

$$A_1 = \frac{1}{2\zeta}(P - P''), \tag{5.63}$$

which enables us to transform (5.61) and (5.62) to the sought form, namely

$$A_2 = -\tfrac{3}{4}\eta P'', \tag{5.64}$$

$$F = \frac{\eta}{4\zeta}[(2\zeta^2 - 1)P'' - 4\zeta P' + P] - W. \tag{5.65}$$

Thus expressions (5.63)–(5.65) together with (5.54)–(5.57) determine the free functions in the general solution (5.41)–(5.43) in terms of the given boundary distributions of velocity (5.53) and pressure (5.54). So we can next investigate which distribution of pressure perturbation resolves the singularity in spine reconnection driven by ideal MHD flows. For this one needs to write the general solution (5.41)–(5.46) in terms of the imposed boundary conditions. This can be done by substituting (5.63)–(5.65) into (5.54)–(5.57) and then the results into the solution (5.41)–(5.46) to give

$$j_R = \frac{R(P - P'')}{2\zeta}\cos\phi, \tag{5.66}$$

$$j_\phi = -RP'\sin\phi, \tag{5.67}$$

$$j_z = \frac{P''}{R^2}\cos\phi, \tag{5.68}$$

$$v_R = \eta(2R^2 + 1)(R^2 - 1)^2\frac{P''}{4R^3}\sin\phi, \tag{5.69}$$

$$v_\phi = \eta\frac{(R^2 - 1)}{4\zeta^2 R^3}[R^4(\zeta P' - P) + (R^4 + 2R^2\zeta^2 + 2\zeta^2)(P'' - \zeta P''')]\cos\phi, \tag{5.70}$$

$$v_z = \left\{W - \eta\frac{(R^2 - 1)}{4\zeta}[-P + 4\zeta P' + (1 + 4\zeta^2)P'']\right\}\frac{\sin\phi}{R^2}. \tag{5.71}$$

One then requires that all these values be finite as $R \to 0$ or $\zeta = R^2 z \to 0$ for z in a fixed finite range.

So, to find restrictions on the free distribution of $P(\zeta)$ under which the solution is regular, it is necessary to expand $P(\zeta)$ in a Taylor series near $\zeta = 0$

$$P(\zeta) = P_0 + \zeta P_0' + \tfrac{1}{2}\zeta^2 P_0'' + \tfrac{1}{6}\zeta^3 P_0''' + \dots$$

and substitute into (5.66)–(5.71). An analysis of the resulting expressions shows that all the physical quantities (5.66)–(5.71) are regular near $R = 0$ if the following conditions

$$P_0' = -4W/3\eta, \tag{5.72}$$

$$P_0 = P_0'' = P_0''' = 0 \tag{5.73}$$

are fulfilled. These ensure finiteness of the quantities as $R \to 0$, but they still admit in principle a discontinuity there for some of them. It turns out that one of the quantities, namely v_z, is really discontinuous at the spine. Its expansion near $R = 0$ yields

$$v_z = (W + \tfrac{1}{8}\eta P_0^{\mathrm{IV}} z)\sin\phi,$$

so that at fixed $\phi \neq 0$ we have a discontinuity

$$[v_z] = 2(W + \tfrac{1}{8}\eta P_0^{\mathrm{IV}} z)\sin\phi, \tag{5.74}$$

which cannot be avoided in any way (except by taking $W = P_0^{\mathrm{IV}} = 0$, which corresponds to a trivial 'non-reconnective' solution).

This result is in good agreement with our 'antireconnection' theorem which prohibits convective flows across the fan in analytic solutions of the problem. The above solution explicitly demonstrates that the effects of a magnetostatic pressure perturbation in combination with resistivity may smooth the 'spine' singularity only within such a discontinuity. In this situation viscosity may be important for a subsequent resolution of the discontinuity on a small length scale determined by the Reynolds number. However, such a task is outside the scope of this paper.

Consider now the properties of the above solution in more detail. Choose the function $P(\zeta)$ in the form (see figure 7*a*)

$$P(\zeta) = \frac{-4W\zeta}{3\eta(1 + \zeta^4/\zeta_0^4)}, \tag{5.75}$$

which is the simplest rational function satisfying the conditions (5.72) and (5.73). It is odd and has $P_0^{\mathrm{IV}} = 0$, so that the spine discontinuity (5.74) in this case is completely due to $W \neq 0$. A non-symmetric distribution $P(\zeta)$ would represent a more general solution, but we can expect that, in spite of its symmetry, (5.75) possesses the most characteristic property of the above solutions, namely the presence of both a compression and expansion of plasma near the fan plane (figure 7*a*). This effect is inevitable due to the conditions of regularity (5.72) and (5.73) and is represented most generally by a non-symmetrical distribution $P(\zeta)$.

As $\nabla \cdot \boldsymbol{j} = 0$, there must exist, at least locally, Euler potentials for the current density $\boldsymbol{j}$; it happens that in our case they can be found explicitly, namely,

$$\boldsymbol{j} = \nabla p \times \nabla q, \tag{5.76}$$

where p is simply the pressure given by (5.54) and the other potential is

$$q = \frac{1}{2}\left(\ln|z| - \int_0^{\zeta} \frac{P''\,\mathrm{d}\zeta}{\zeta P}\right). \tag{5.77}$$

For our example (5.75), we have in particular

$$q = \tfrac{1}{2}\ln|z| + \frac{4\zeta^2}{\zeta^4 + \zeta_0^4} + \frac{1}{\zeta_0^2}\arctan\left(\frac{\zeta^2}{\zeta_0^2}\right). \tag{5.78}$$

Note that q does not depend on ϕ, so that the surfaces of constant q are axisymmetric and therefore may be obtained by revolving the contours $q = \text{const.}$ in the (x, z)-plane around the z-axis (see figure 7*b*). The intersections of the two families of surfaces, where $p = \text{const.}$ and $q = \text{const.}$, determine the current lines, to which the current density $\boldsymbol{j}$ is tangent. Taking into account also that the magnetic field $\boldsymbol{B}_0$ is normal to the gradient of p, we have the well-known fact for magnetostatic configurations that the surfaces of constant pressure are 'woven' from the current and magnetic field lines. In our case these surfaces look like strongly deformed ropes, in which the plasma is confined by the Lorentz force $\boldsymbol{j} \times \boldsymbol{B}_0$ (figure 8*a*). As is seen from figure 8*b* showing the cross-sections of such surfaces with the plane $z = 1$, one can expect that their form is similar to that of the corresponding current and field lines.

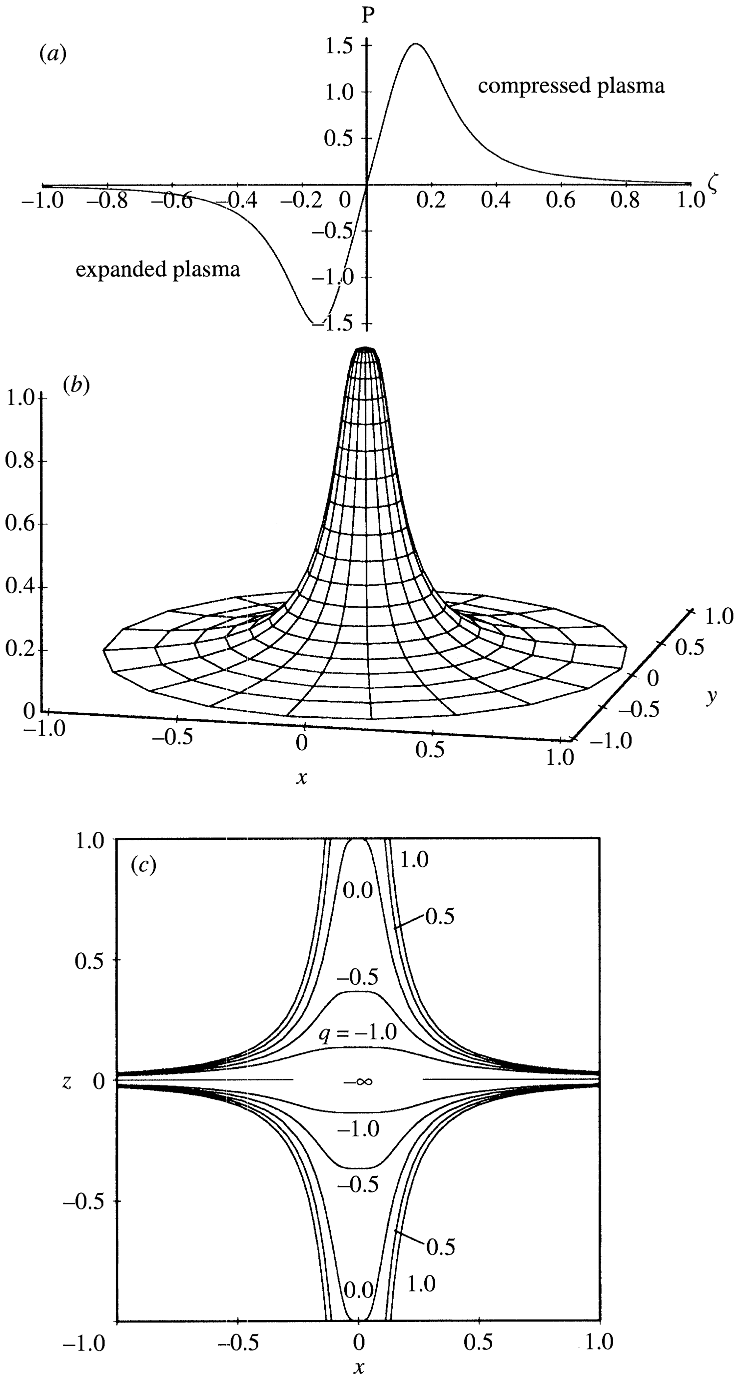

Figure 7. (*a*) The variation of the pressure amplitude $P(\zeta)$ with $\zeta = z^2R$ for $W/\eta = -10$ and $\zeta_0 = 0.2$. It resolves a singularity of the ideal MHD flow when a non-vanishing plasma resistivity is included. (*b*) The axisymmetric shape of the surface on which the Euler potential q (5.78) for the current density (5.76) vanishes, using the function $P(\zeta)$ shown in figure 7*a*. All the other surfaces of constant q have similar shapes. (*c*) Their cross sections with the plane passing through the axis of symmetry $R = 0$.

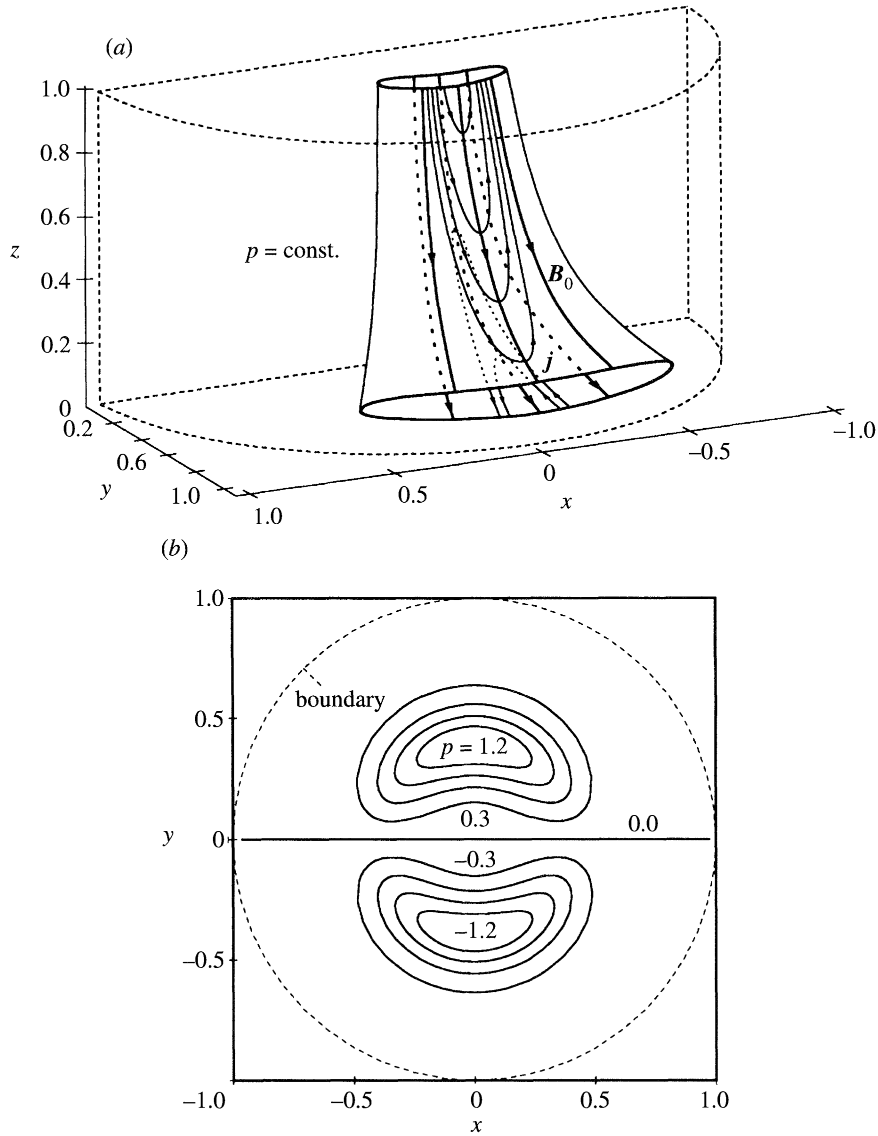

Figure 8. (*a*) The surface of constant pressure p in one quarter of the cylindrical volume; the thin and thick lines here are, respectively, current and magnetic field lines lying on this surface. (*b*) The cross sections of the surfaces where $p = \text{const.}$ with the plane $z = 1$. The function $P(\zeta)$ used here is shown in figure 7*a*.

Due to the symmetry of the solution, the isobaric surfaces and the corresponding current and field lines on them in the other quadrants of space are mirror images, both about the vertical (x, z)-plane and also about the horizontal (x, y)-plane; the appropriate values of pressure perturbation here reverse their signs at each reflection.

Let us discuss now the properties of the plasma flow described by this solution. First of all, for the velocity as for the current density, due to $\nabla \cdot \boldsymbol{v} = 0$, there must exist Euler potentials, which, however, we have not been able to find explicitly. Nevertheless, it is not difficult to see that one of these potentials may be determined

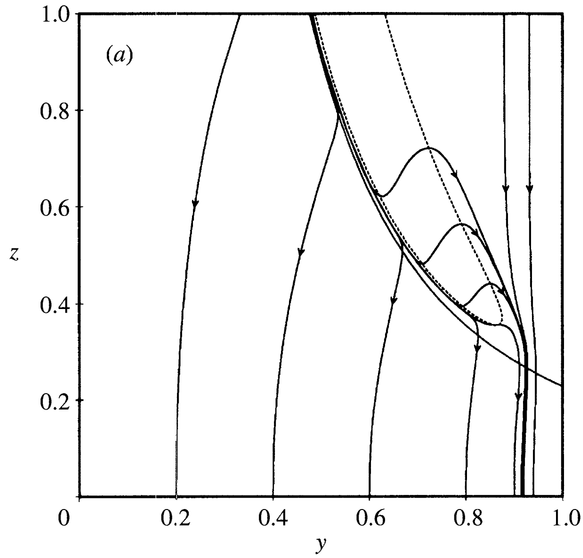

Figure 9. (*a*) The streamlines of the velocity field (5.69)–(5.71) in the plane $\phi = \pi/2$; here the dashed and dotted lines represent the lines where, respectively, v_R and v_z vanish. The function $P(\zeta)$ entering into the expressions for v_R and v_z is shown in figure 7*a*.

as a function of z and R only, since in one of the streamline equations, namely

$$\frac{\mathrm{d}R}{v_R} = \frac{\mathrm{d}z}{v_z},$$

where v_R and v_z are determined by (5.69) and (5.71), there is no dependence on ϕ. So each streamline of the velocity field (5.69)–(5.71) lies on a surface of revolution about the z-axis. The cross-sections of these surfaces with a plane passing through the z-axis coincide with the streamlines lying in the (y, z)-plane where $v_\phi \equiv 0$.

Such streamlines are shown in figure 9*a* in the first quadrant of this plane. The picture of the flow in the other quadrants is symmetrical: the streamlines cross the fan plane and continue below it, symmetrically reproducing there the form they have above the fan, while on the left of the axis of symmetry we have the same picture as on the right but with the opposite direction for the velocity.

The characteristic feature of the above flow is the presence in it of very strong jets of plasma, which diverge from the regions of enhanced pressure (cf. figures 8 and 9) and then, after crossing the fan, they converge again in the regions of reduced pressure. The jet in each of these regions is so strong that the streamlines look in figure 9*a* as if they start from the same line. This fact is clarified in figure 9*b* which shows that near this jet the absolute value of velocity is increased by more than two orders of magnitude. Such a behaviour of the flow is probably caused by the dominating diffusion effect due to the pressure gradient in (5.36) – under the action of this gradient the plasma is simply squeezed out from the high-pressure regions.

This effect for the azimuthal component of velocity v_ϕ dominates, however, only in the closest neighbourhood of the pressure maximum (see figure 9*c*). In the region far from this maximum and closer to the axis of flow the drift velocity term (the first one in (5.36)) becomes more important, forming thereby a shearing motion of plasma in the azimuthal direction near the plasma ropes.

As the resistivity (η) of the plasma decreases or its velocity (W) on the boundary

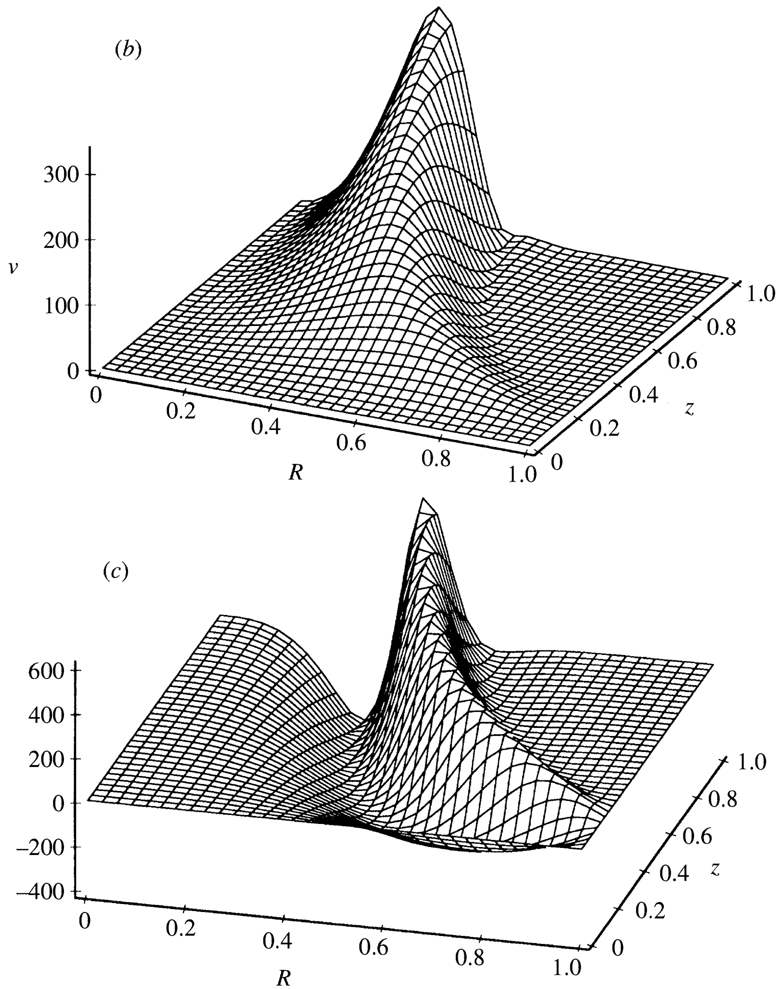

Figure 9. (*b*) The distribution of the speed $|\boldsymbol{v}| = (v_R^2 + v_z^2)^{1/2}$ in the plane $\phi = \pi/2$ and (*c*) the distribution of v_ϕ in the plane $\phi = 0$ for the velocity field (5.69)–(5.71).

increases, the amplitudes of all the physical values grow in the solution. The scale of the whole process is controlled by the free parameter ζ_0 – the smaller its value, the more compact a structure is responsible for resolving the ideal MHD singularity. Although, as is seen from the discussion of the flow, this singularity is resolved by resistivity and a pressure perturbation only to within a tangential discontinuity of velocity at the axis of symmetry of the flow.

(iii) *Spine reconnection: solution for $m \geqslant 1$ and general boundary conditions*

We have considered above the simplest resistive solution for spine reconnection, where the boundary conditions are very special, namely, they imply only a vertical and homogeneous motion of plasma along the z-axis on the boundary. This particular kind of boundary conditions leads to a discontinuity of v_z at the spine, while all the other values are regular and continuous everywhere in the volume. It is interesting therefore to investigate whether such a result is preserved for more general conditions.

Consider this question in the most general form by taking the boundary velocity as

$$v_{bR} = U(\zeta)\sin(m\phi), \tag{5.79}$$

$$v_{b\phi} = V(\zeta)\cos(m\phi), \tag{5.80}$$

$$v_{bz} = W(\zeta)\sin(m\phi), \tag{5.81}$$

where the integer number $m \geqslant 1$ marks the corresponding mode. Generally its amplitude depends on m, so that the arbitrary functions $U(\zeta)$, $V(\zeta)$ and $W(\zeta)$ should be written with subscript m, which we omit for brevity.

The unknown values may be specified as in (5.54)–(5.57) by replacing ϕ with $m\phi$ and then after substituting these expressions into (5.47)–(5.49) we find

$$A_2 + \tfrac{1}{4}\eta(-2m\zeta A_1 + 2P'' + m^2 P) = U, \tag{5.82}$$

$$-F' + \tfrac{1}{2}\eta[(1-2\zeta^2)A_1' + m\zeta P' - mP] = V, \tag{5.83}$$

$$-mF - 2\zeta A_2 + \eta(\tfrac{1}{2}mA_1 - P' - \zeta P'') = W. \tag{5.84}$$

These equations can be solved in the same way as (5.58)–(5.60) with the result that

$$A_1 = \frac{m^2 P - P''}{2m\zeta} - \frac{2(\zeta U' + U) - mV + W'}{2m\eta\zeta}, \tag{5.85}$$

$$A_2 = -\tfrac{3}{4}\eta P'' + \tfrac{1}{4}[2(U - \zeta U') + mV - W'], \tag{5.86}$$

$$\begin{aligned} F = {} & \frac{\eta}{4m\zeta}[(2\zeta^2 - 1)P'' - 4\zeta P' + m^2 P] - \frac{1}{4m\zeta}(2U - mV + W') \\ & - \frac{1}{2m}[2\zeta U - (2\zeta^2 - 1)U' + m\zeta V - \zeta W' + 2W], \end{aligned} \tag{5.87}$$

Substituting now (5.85)–(5.87) (with ϕ replaced by $m\phi$) into the general solution (5.41)–(5.46) we obtain

$$j_R = RA_1 \cos m\phi, \tag{5.88}$$

$$j_\phi = -RP' \sin m\phi, \tag{5.89}$$

$$j_z = \frac{1}{R^2}(mP - 2\zeta A_1)\cos m\phi, \tag{5.90}$$

$$v_R = \left[RA_2 - \frac{\eta}{4R^3}\left(2m\zeta A_1 - 2R^6 P'' - m^2 P\right)\right]\sin m\phi, \tag{5.91}$$

$$v_\phi = -\left\{RF' + \frac{\eta}{2R^3}\left[(2\zeta^2 - R^6)A_1' - m\zeta P' + mP\right]\right\}\cos m\phi, \tag{5.92}$$

$$v_z = \left[-\frac{1}{R^2}(2\zeta A_2 + mF) + \frac{\eta}{2}(mA_1 - 2\zeta P'' - 2P')\right]\sin m\phi. \tag{5.93}$$

These formulae together with (5.85)–(5.87) represent the solution for an arbitrary mode ($m \geqslant 1$) satisfying the general boundary conditions (5.79)–(5.81) of the spine reconnection process.

For an arbitrary distribution of $P(\zeta)$ the resulting solution is generally singular, with the current density ($\boldsymbol{j}$) and velocity ($\boldsymbol{v}$) tending to infinity as $R \to 0$. So let us find what restrictions should be imposed on $P(\zeta)$ to obtain regular behaviour. Expand all the functions of ζ in (5.85)–(5.93) in a Taylor series about $\zeta = 0$, supposing that all of them are analytical and denoting the values at $\zeta = 0$ by zero subscripts.

Then the regularity condition for j_R is

$$m^2 P_0 - P_0'' - (2U_0 - mV_0 + W_0')/\eta = 0. \tag{5.94}$$

It is not difficult to see that this condition ensures also the finiteness of $F(\zeta)$ (see (5.87)). The component j_ϕ is automatically regular for an analytical $P(\zeta)$, while the regularity of j_z (or v_R) requires two extra conditions, namely

$$P_0 = 0, \tag{5.95}$$

$$mP_0' - 2A_{10} = 0, \qquad 2A_{10} = mP_0' - \frac{P_0'''}{m} - \frac{1}{m\eta}(4U_0' - mV_0' + W_0''), \tag{5.96}$$

The regularity of v_ϕ requires only (5.95) and v_z is regular provided

$$F_0 = 0, \qquad F_0 = -\{\eta[P_0''' + (4 - m^2)P_0'] + 4U_0' - mV_0' + W_0'' + 4W_0\}/(4m). \tag{5.97}$$

If $m \neq 2$, the resulting conditions enable us to determine P_0 from (5.95) and the derivatives

$$P_0' = \frac{4W_0}{(m^2 - 4)\eta}, \tag{5.98}$$

$$P_0'' = (-2U_0 + mV_0 - W_0')/\eta, \tag{5.99}$$

$$P_0''' = (-4U_0' + mV_0' - W_0'')/\eta, \tag{5.100}$$

in terms of the local characteristics of the boundary velocity near the fan plane. Thus the requirement of regularity fixes the first four terms in the Taylor expansion of $P(\zeta)$ at $\zeta = 0$, while all the other terms remain arbitrary – this general result is similar to (5.72)–(5.73) obtained for the first mode and the particular boundary conditions.

The same analysis of regularity for the second mode $m = 2$ yields

$$W_0 = 0, \tag{5.101}$$

instead of (5.98) and the expressions (5.99) and (5.100) (with $m = 2$) for P_0'' and P_0''', while P_0' becomes arbitrary in this case. It is easy to see from (5.93) that at the fan plane

$$v_z|_{z=0} = \tfrac{1}{2}\eta(mA_{10} - 2P_0')\sin m\phi,$$

or, taking into account (5.96) and (5.97), for all the modes

$$v_z|_{z=0} = W_0 \sin m\phi. \tag{5.102}$$

So the condition (5.102) implies that the second mode is regular only if there is no convective flow across the fan plane. It also shows that for all the other modes ($m \geqslant 1$) there is a discontinuity in v_z at $R = z = 0$ if (5.101) is not satisfied. Thus we arrive again at the result consistent with our 'antireconnection' theorem: namely, that regular and continuous plasma flows crossing the fan plane of the null point are impossible if nonlinear and viscous effects are negligible.

To be precise, one should add that generally the condition (5.101) is not sufficient for continuity of v_z, since from (5.93) its value at the spine is

$$v_z|_{R=0} = [\tfrac{1}{2}\eta(mA_{10} - 2P_0') - (mF_0' + 2A_{20})z]\sin m\phi.$$

Therefore, in addition to (5.96) or (5.101), one needs to require

$$mF_0' + 2A_{20} = 0 \tag{5.103}$$

for v_z to be continuous at the spine. From (5.86) and (5.87) we have

$$A_{20} = -\tfrac{3}{4}\eta P_0'' + \tfrac{1}{4}(2U_0 + mV_0 - W_0'),$$

$$F_0' = -\{\eta[P_0^{\mathrm{IV}} + (4 - m^2)P_0''] + 8U_0 + 6U_0'' + m(4V_0 - V_0'') + 4W_0' + W_0'''\}/(8m),$$

which together with (5.99) enable us to consider (5.103) as a condition fixing P_0^{IV}; after simple calculations it gives

$$P_0^{\mathrm{IV}} = [2(16-m^2)U_0 - 6U_0'' + m(m^2-16)V_0 + mV_0'' + (8-m^2)W_0' - W_0''']/\eta. \quad (5.104)$$

(iv) *Spine reconnection: solution for an axisymmetric mode* $m = 0$

The most general axisymmetric solution may be obtained from formulae (5.41)–(5.46) if one omits their dependence on ϕ, so that all the derivatives $\partial/\partial\phi$ vanish there to yield

$$j_R = R\alpha, \quad (5.105)$$

$$j_\phi = -Rp', \quad (5.106)$$

$$j_z = -2z\alpha, \quad (5.107)$$

$$v_R = Ra + \tfrac{1}{2}\eta R^3 p'', \quad (5.108)$$

$$v_\phi = -R\Phi_0' + \tfrac{1}{2}\eta R(R^2 - 2z^2)\alpha', \quad (5.109)$$

$$v_z = -2za - \eta(p' + \zeta p''), \quad (5.110)$$

where the functions α, a, p and Φ_0 depend only on ζ. It is clearly seen from here that the axisymmetric solution has no singularities provided these functions are regular.

Equating now the right-hand sides of (5.108)–(5.110) at $R = 1$ with its boundary values

$$v_{bR} \equiv U(\zeta), \quad (5.111)$$

$$v_{b\phi} \equiv V(\zeta), \quad (5.112)$$

$$v_{bz} \equiv W(\zeta), \quad (5.113)$$

one can easily express three of the indicated four functions in terms of these values to obtain

$$a = 2U + \zeta U' + \tfrac{1}{2}W', \quad (5.114)$$

$$p = -\frac{1}{\eta}\int (2\zeta U + W)\,\mathrm{d}\zeta, \quad (5.115)$$

$$\Phi_0 = \int [\tfrac{1}{2}\eta(1 - 2\zeta^2)\alpha' - V]\,\mathrm{d}\zeta. \quad (5.116)$$

Notice that, contrary to the cases of $m \geqslant 1$, for the axisymmetric mode we can no longer take the pressure perturbation $p(\zeta)$ as a free function, determined independently of the boundary velocity components $U(\zeta)$ and $W(\zeta)$ – according to (5.115), fixing $U(\zeta)$ and $W(\zeta)$ unambiguously determines the form of $p(\zeta)$. One can take, however, in this case $\alpha(\zeta)$ as a free function for our axisymmetric boundary-value problem. The substitution of (5.114)–(5.116) into (5.105)–(5.110) gives us the general solution of this problem, in which j_R and j_z are given by (5.105) and (5.107), while

$$j_\phi = R(2\zeta U + W)/\eta, \quad (5.117)$$

$$v_R = R[(2-R^2)U + (1-R^2)(\zeta U' + \tfrac{1}{2}W')], \tag{5.118}$$

$$v_\phi = RV + \tfrac{1}{2}\eta R(R^2-1)[1 + 2z^2(R^2+1)\alpha'], \tag{5.119}$$

$$v_z = W + z(R^2-1)(4U + 2\zeta U' + W'). \tag{5.120}$$

The plasma flow is represented here as linear superposition of several flows caused by different physical effects; for example, the rotation of plasma around the z-axis (see (5.119)) is generally due to, first, the azimuthal component of velocity at the boundary (5.112) and, second, a non-vanishing $\alpha(\zeta)$ or, in other words, because of the presence of longitudinal currents $j_{\|}$ in the configuration. The first effect does not depend on the plasma resistivity η, since it is caused by the corresponding plasma movement prescribed at the boundary $R = 1$ and reduced to a purely convective flow in the limit $\eta \to 0$. In contrast, the second effect disappears in this limit, because it results from a plasma drift produced by the magnetic field $\boldsymbol{B}_0$ and electric field in the plane $\phi = \text{const.}$; the latter appears here to support the longitudinal currents in a resistive plasma.

In the first case the plasma particles, passing through a magnetic line characterized by some ζ, move together as a solid body with a constant angular velocity $V(\zeta)$, so that in the limit $\eta \to 0$ one can consider them be 'frozen' to the magnetic field lines, the footpoints of which rotate at the boundary $R = 1$ with the velocity $V(\zeta)$. Therefore, this flow can be interpreted as purely convective and caused by the appropriate movement of the magnetic field lines. In the second case the plasma particles 'slip off' the field lines, since their angular velocity is no longer constant at these lines. Such a component of plasma motion is purely diffusive and caused by the resistive dissipation of the longitudinal currents.

From the viewpoint of our 'antireconnection' theorem it is interesting now to study the component of plasma flow in the (R, z)-plane. So let us exclude from consideration the azimuthal flow by putting

$$V(\zeta) \equiv 0, \qquad \alpha(\zeta) \equiv 0.$$

Then the velocity of the flow has only v_R and v_z components determined by (5.118) and (5.120), which can be expressed in terms of the stream function Ψ (see § 3 *a*). One can derive from (5.118) and (5.120) that

$$R\Psi = (R^2-1)(\zeta U + \tfrac{1}{2}W) - \int U \, \mathrm{d}\zeta. \tag{5.121}$$

Although the resistivity η does not enter into this expression explicitly, the flow across the magnetic field lines is ultimately due to Ohmic dissipation of the azimuthal current j_ϕ appearing here as a result of the pressure perturbation. This fact is clearly expressed by formulae (5.106), (5.108), (5.110) as well as by (5.115)–(5.120).

To consider the effect of the flow across the fan plane in a pure form, let us put the radial velocity of the flow at the boundary $R = 1$ equal to zero, i.e. $U \equiv 0$. Then the desired flow across the fan can be set up by choosing the boundary z-component of velocity as

$$W = -\frac{1}{1 + \zeta^2/\zeta_0^2}, \tag{5.122}$$

the absolute value of which vanishes at infinity and has a maximum of width $\sim \zeta_0$ at the fan. According to (5.115) such a boundary flow is supported by a pressure

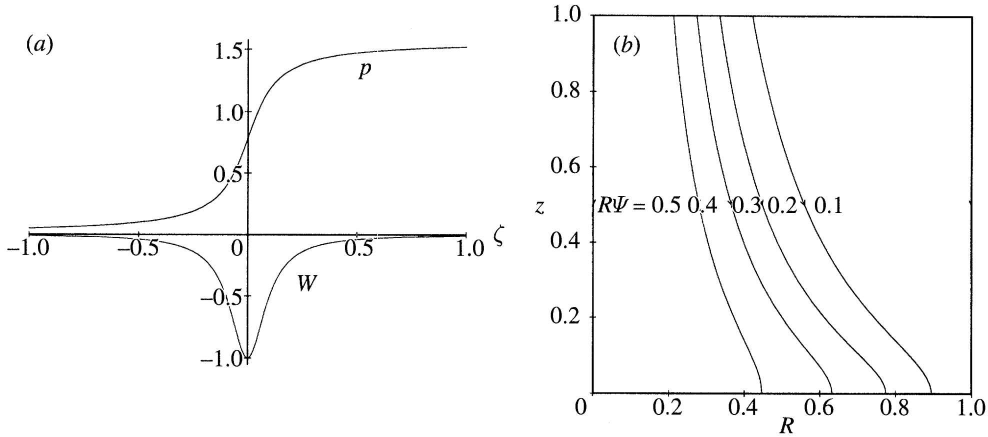

Figure 10. (*a*) The distributions at the boundary $R = 1$ (so that $z = \zeta$) of the z-component of velocity $W(\zeta)$ (165) and the pressure perturbation $p(\zeta)$ (166) which determine a regular and continuous diffusive flow across the fan plane. (*b*) The corresponding streamlines of the flow in the first quadrant of the plane passing through the axis of symmetry.

distribution

$$p = \frac{z0}{\eta}\left[\frac{\pi}{2} + \arctan\left(\frac{\zeta}{\zeta_0}\right)\right] \tag{5.123}$$

describing a decrease of pressure from the upper to the lower half space (see figure 10*a*). The corresponding gradient of pressure squeezes the plasma in this direction, thereby causing it to flow across the fan plane (figure 10*b*). As the electric potential (5.38) in the configuration is identically zero, this flow is not convective in nature – one can see that it is completely diffusive and is due to the Ohmic dissipation of the azimuthal current j_ϕ concentrated near the fan plane (figure 11*a*). The interaction of such a current with the magnetic field $\boldsymbol{B}_0$ results in a Lorentz force counterbalancing the pressure gradient (figure 11*b*). This example clearly demonstrates that it is not difficult to create a regular magnetostatic perturbation of pressure near the null point configuration with a regular and continuous plasma flow crossing the fan plane, but this flow will be diffusive in nature rather than convective – exactly, as our 'antireconnection' theorem predicts.

6. Configurations of two isolated null points: separator reconnection

Let us now investigate some of the kinematic properties of configurations containing two null points, say at $z = \pm 1$ on the z-axis. In general the flows and electric fields in such configurations are exceedingly complex (Lau & Finn 1990), but analytical progress may be made by considering the following form for the field components

$$B_x = xf(z), \quad B_y = yg(z), \quad B_z = 1 - z^2, \tag{6.1}$$

where the condition $\nabla \cdot \boldsymbol{B} = 0$ implies that the functions $f(z)$ and $g(z)$ are related by

$$f(z) + g(z) = 2z. \tag{6.2}$$

The special field line joining the two null points is known as a *separator* and it is the z-axis. Three distinct cases arise depending on the nature of the separator near the null points. These are when the separator is:

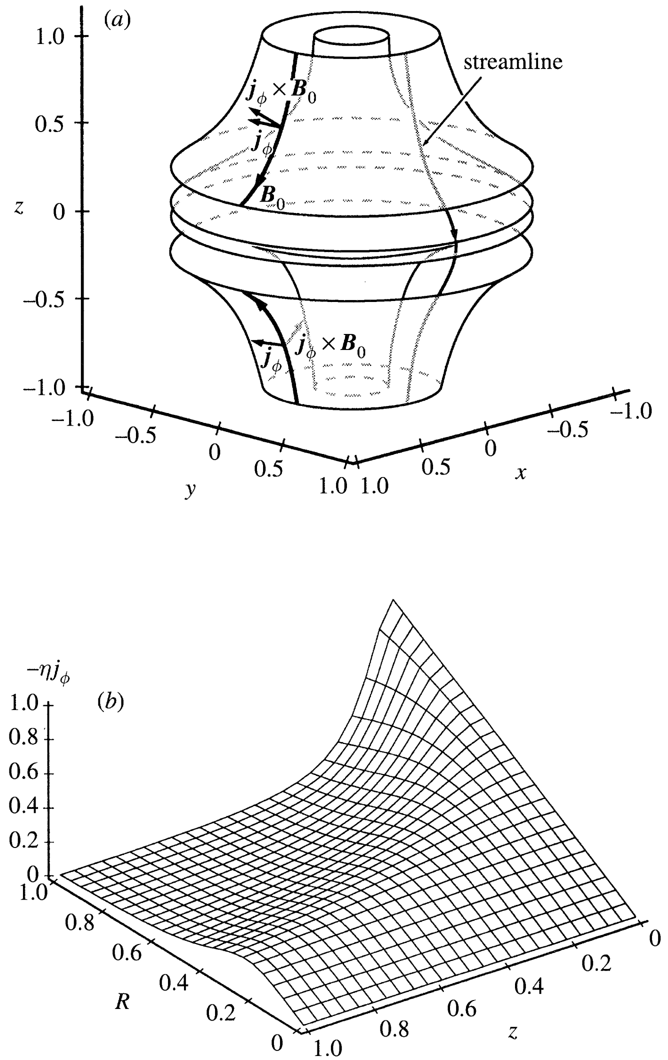

Figure 11. (a) The surfaces of constant pressure distribution with the radial gradient of pressure directed towards and away from the axis of symmetry, respectively, above and below the fan plane for the example of $p(\zeta)$ shown in figure 10a. (b) The corresponding distribution of the azimuthal current j_ϕ in the first quadrant of the plane passing through the axis of symmetry.

(i) the spine of both null points;
(ii) the spine of one null point and part of the fan of the other;
(iii) part of the fans of both null points.

Case (iii) is the generic case since there is a continuum of field lines lying in the fans whereas only single field lines comprise the spines. Thus in general cases (i) and (ii) are not structurally stable: if the null points are moved or rotated, the topology in figures 12a and 12b will not persist. For example, if the lower null in figure 12a is displaced in the x-direction, the spines will diverge from one another to left and right and will no longer be a separator joining the nulls.

By taking different forms for $f(z)$ we may set up configurations with different types and orientations of null points: for the purpose of studying kinematic properties

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

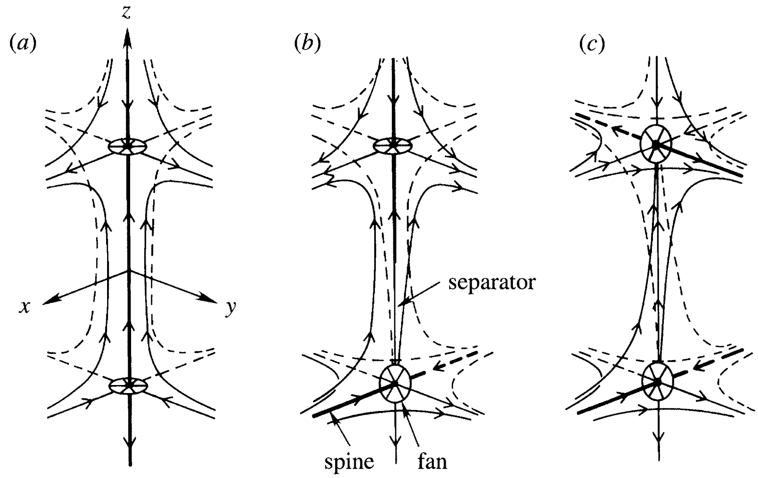

Figure 12. Magnetic fields of two null points at $z = \pm 1$ on the z-axis, joined by a separator and showing the spines (thick lines) and fans (circles) of each null when the separatrix is (*a*) the spine of both null points (*b*) the spine of only one and (*c*) the spine of neither.

these have the great advantage of analytical simplicity, but they are not in general magnetostatic in nature. In general they will be improper radial or spiral nulls, but by careful choice we may set up configurations of two proper radial nulls, which have the advantage that we have already investigated in detail some of the behaviour near such nulls in the previous sections.

As an example of case (i) we may choose $f(z) = g(z) = z$, so that

$$(B_x, B_y, B_z) = (xz, yz, 1 - z^2), \tag{6.3}$$

which has field lines given by

$$x^2(1 - z^2) = c, \qquad y = kx, \tag{6.4}$$

where c and k are constants. The field lines lie in planes $y = kx$ and the basic structure of the two nulls is identical since the field near $z = 1$ is $(B_x, B_y, B_z) \approx (x, y, -2(z-1))$ whereas the field near $z = -1$ is $(B_x, B_y, B_z) \approx -(x, y, -2(z+1))$. Both nulls have the z-axis as spine and planes parallel to the xy-plane as fans (figure 12*a*).

As an example of case (ii), set $f(z) = \frac{1}{2}(5z - 3)$, so that

$$(B_x, B_y, B_z) = (\tfrac{1}{2}x(5z - 3), \tfrac{1}{2}y(3 - z), 1 - z^2). \tag{6.5}$$

and the field lines are given by

$$x^2(z - 1)(z + 1)^4 = c, \qquad y^2(z - 1)(z + 1)^{-2} = k. \tag{6.6}$$

Near $z = 1$, the field is $(B_x, B_y, B_z) \approx (x, y, -2(z-1))$ and so the null point is identical to the one we have studied in § 4, except that it is located at $z = 1$ on the z-axis rather than at the origin. The spine of this null is the z-axis and the fan is parallel to the xy-plane (figure 12*b*). In contrast, near $z = 1$, the field is $(B_x, B_y, B_z) \approx 2(-2x, y, z + 1)$, so that the spine and fan are parallel to the x-axis and yz-plane, respectively.

Case (iii) may be exemplified by putting $f(z) = z - 3$, so that

$$(B_x, B_y, B_z) = (x(z - 3), y(z + 3), 1 - z^2), \tag{6.7}$$

for which the field lines are now

$$x(z+1)^2(z-1)^{-1} = c, \qquad y(z-1)^2(z+1)^{-1} = k. \tag{6.8}$$

Near $z = 1$, the field becomes $(B_x, B_y, B_z) \approx -2(x, -2y, z-1)$ and so the spine and fan are aligned with the y-axis and xz-plane, respectively (figure 12*c*). Near the null point ($z = -1$), the field approximates to $(B_x, B_y, B_z) = 2(-2x, y, z+1)$ and the spine and fan are therefore parallel to the x-axis and yz-plane.

Now let us ask: what are the electric field ($\boldsymbol{E}$) and field line velocity ($\boldsymbol{v}_\perp$) for steady kinematic reconnection? Since the motion is steady, $\nabla \times \boldsymbol{E} = 0$ and so

$$\boldsymbol{E} = \nabla \Phi.$$

Furthermore, in the ideal region

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0, \tag{6.9}$$

and so the electric field is perpendicular to the magnetic field or

$$\boldsymbol{B} \cdot \nabla \Phi = 0.$$

Thus, the electric potential Φ is constant along field lines; in other words, in the above examples, since constant values of c and k describe the field lines,

$$\Phi = \Phi(c, k).$$

Many different functional forms of Φ may be considered, depending on the motions of the footpoints and the nature of the reconnection. But, once it has been determined, the electric field components follow as

$$E_x = \frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial c}\frac{\partial c}{\partial x} + \frac{\partial \Phi}{\partial k}\frac{\partial k}{\partial x},$$

$$E_y = \frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial c}\frac{\partial c}{\partial y} + \frac{\partial \Phi}{\partial k}\frac{\partial k}{\partial y},$$

$$E_z = \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial c}\frac{\partial c}{\partial z} + \frac{\partial \Phi}{\partial k}\frac{\partial k}{\partial z},$$

where the derivatives of c and k are given functions for a particular given configuration. Then the field line velocity follows from (6.9) as

$$\boldsymbol{v}_\perp = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}.$$

As a particular example, suppose that spine reconnection is being driven at the upper null point of the field (6.7) with $\Phi = v_{\mathrm{e}} \cos \tilde{\phi}$ (so that $E_{\tilde{\phi}} = v_{\mathrm{e}} \sin \tilde{\phi}$, where $\tilde{\phi}$ is the azimuthal angle in cylindrical coordinates with respect to the upper spine as axis. Then near the null at $z = 1$ the potential may be written as

$$\Phi = \frac{v_{\mathrm{e}}}{\surd(1 + (z-1)^2/x^2)}, \tag{6.10}$$

where (6.8) implies that $c \approx 4x/(z-1)$ and $k \approx \frac{1}{2}y(z-1)^2$. Thus the potential (6.10) may be written in the required form in terms of c and k as simply

$$\Phi = \frac{1}{\surd(1 + 16/c^2)},$$

which is valid throughout the domain and not just near the upper null point. The

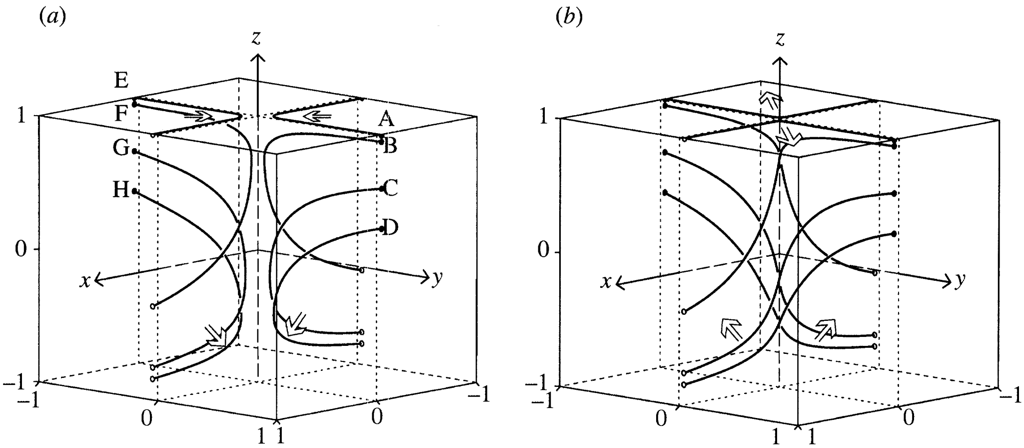

Figure 13. Separator reconnection driven by horizontal footpoint motions on the vertical planes $y = \pm 1$.

resulting forms for $\boldsymbol{E}$ and $\boldsymbol{v}_\perp$ everywhere may therefore be calculated analytically, and in particular we find that near the lower null point

$$(E_x, E_y, E_z) = ((z+1)^2, 0, -x)/8,$$

$$(v_{\perp x}, v_{\perp y}, v_{\perp z}) = \frac{1}{4B^2}(xy, 2x^2 - (z+1)^3, y(z+1)^2),$$

where $B^2 = 4(4x^2 + y^2 + (z+1)^2)$.

Just as different kinds of reconnection can be driven at a single null point, so the kind of reconnection that occurs at a pair of null points depends on the nature of the imposed flows. For example, suppose we impose the motions of footpoints on the vertical planes $y = \pm 1$ in figure 12*c*. In particular, move the footpoints with a purely horizontal velocity (v_x) from right to left on $y = 1$ and left to right on $y = -1$ (figure 13). Such footpoints cross the spine of the upper null point and so drive fan reconection at the upper null: they also cross the fan ($x = 0$) of the lower null point and so drive spine reconnection at the lower null. This may be analysed as follows. Suppose the footpoints are at the points $(x_0, 1, z_0)$ on the plane $y = 1$. Then according to (6.8) the constants (c, k) are given in terms of x_0 and z_0 by

$$c = \frac{x_0(z_0+1)^2}{z_0 - 1}, \quad k = \frac{(z_0-1)^2}{z_0+1}. \tag{6.11}$$

When $x_0 < 0$, the field lines from these feet are given by (6.8) and meet the plane $x = -1$ at $(-1, y, z)$ where y and z are given by

$$\frac{(z+1)^2}{z-1} = -x_0\frac{(z_0+1)^2}{z_0-1}, \quad y = \frac{(z+1)(z_0-1)^2}{(z-1)^2(z_0+1)}. \tag{6.12}$$

Thus as x_0 approaches 0 and the footpoints cross the fan of the lower null point, so z approaches -1 and y tends to zero: in other words, the other end of the field line crosses the spine of the lower null. For $x_0 = 0$, (6.11) implies that $c = 0$ and so from (6.8) the field lines lie in the plane $x = 0$ with equations

$$y = k\frac{(z+1)}{(z-1)^2}.$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

When $x_0 > 0$, the field lines meet the vertical plane $x = 1$ at $(1, y, z)$ where y and z are given by

$$\frac{(z+1)^2}{z-1} = x_0\frac{(z_0+1)^2}{(z_0-1)}, \quad y = \frac{(z+1)(z_0-1)^2}{(z-1)^2(z_0+1)}.$$

Thus the equations of the field lines and the locations of the other ends on the surface of the cube of side 2 can be calculated for various values of z_0 as x_0 decreases.

In order to find Φ (and therefore if required $\boldsymbol{E}$ and $\boldsymbol{v}_\perp$) everywhere, we use the fact that $v_z = 0$ on $y = 1$ to deduce that

$$E_xB_y = E_yB_x,$$

or, if $\Phi = \Phi(c, k)$,

$$\frac{\partial\Phi}{\partial c}\frac{(z_0+1)^2}{(z_0-1)}(z_0+3) = \frac{\partial\Phi}{\partial k}\frac{(z_0-1)^2}{(z_0+1)}x_0(z_0-3),$$

where $z_0 = z_0(k)$ and $x_0(c, k)$ are given by (6.11). It turns out to be easier to regard Φ as a function of x_0 and z_0, so that this equation reduces to

$$\frac{1}{x_0}\frac{\partial\Phi}{\partial x_0} = \frac{(z_0-1)(z_0+1)(z_0-3)}{(z_0+3)^2}\frac{\partial\Phi}{\partial z_0}. \tag{6.13}$$

The equation for the characteristics of (6.13) is

$$\tfrac{1}{2}x_0^2 + \frac{(z_0+1)^{1/2}(z_0-3)^{9/2}}{(z_0-1)^4} = \text{const}.$$

and so the general solution of (6.13) is

$$\Phi = f\left[\tfrac{1}{2}x_0^2 + \frac{(z_0+1)^{1/2}(z_0-3)^{9/2}}{(z_0-1)^4}\right]. \tag{6.14}$$

Then a variety of different functional forms for $f(s)$ may be adopted to represent different types of horizontal flow on the boundary.

The resulting potential throughout the volume is

$$\Phi = f(u),$$

where

$$u = \tfrac{1}{2}x^2 + (z_0+1)^{1/2}(z_0-3)^{9/2}(z_0-1)^{-4}$$

and from (6.11)

$$z_0 = \tfrac{1}{2}\{(k+2) - \surd(k^2+8k)\},$$

while (6.8) implies

$$k = y(z-1)^2/(z+1).$$

However, there are many other types of footpoint motion in addition to those in figure 13. For example, purely vertical motions on the plane $y = 1$ will cross the spine of the upper null point and so will drive fan reconnection there; but they will not cross the fan or spine of the lower null point and so no reconnection will occur at it. On the other hand, a flow pattern on $y = 1$ which is convergent on the upper spine from below and which crosses the fan of the lower null will drive spine reconnection at both null points.

7. Conclusion

We have here considered the physics of magnetic reconnection at a three-dimensional neutral point, focusing on the simplest case of a proper radial null, although it is likely that many of the basic features are robust and will apply to the other types, namely improper radial nulls and spiral nulls. In discussing these features, the important roles of the key elements of the structure of a null have been stressed, namely the spine and the fan.

Three kinds of reconnection have been put forward. In spine reconnection, continuous motions of the footpoints are imposed on any surface (not necessarily cylindrical) that encircles the spine: when these footpoints cross the fan a singular flow is driven all along the spine. For fan reconnection continuous motions are imposed on any two surfaces that cross the fan, one above the null and one below it: when these footpoints cross the spine a singular counter-rotating swirling flow is driven at the fan surface. The third type of reconnection is *separator reconnection* in which field lines in planes perpendicular to a separator (the particular field line in the fan that links to a neighbouring null point) have X-type topology: they may collapse to form a current sheet and so break and reconnect.

An anti-reconnection theorem is proved as an extension to the previous two-dimensional version (Priest *et al.* 1994) although the proof is much more complex. It states essentially that steady linear reconnection is impossible and therefore that the above spine and fan singularities cannot be resolved by magnetic diffusion alone. This statement is illustrated analytically by an explicit general solution describing spine reconnection. It is shown also that the fan plane may be crossed only by diffusive flows in such a situation. The theorem holds both for incompressible and compressible fluids, although the models for spine and fan reconnection are developed only in the incompressible limit; as in two-dimensional reconnection, compressibility is not expected to have a major impact on the nature of reconnection, although it may well be a significant effect in the singular layers where fluid elements are strongly distorted. In future it will be interesting to set up numerical experiments and determine the role of nonlinearity and the way that jets of plasma are accelerated.

Complex configurations containing two null points will in general have the topology shown in figure 10*c*. This is because the two fan surfaces in general intersect in a curve, and this curve must go through both null points since it lies in both fans: in other words, it is a separator. In future we plan to study the role of spines, fans and separators in complex fields which are typical of those on the Sun and which contain many null points.

We are delighted to acknowledge stimulating discussions with Sherry Palmer (who translated our three-dimensional ideas into two-dimensional drawings while dreaming of Skye), Alex Craik, Clare Parnell, Jason Smith, Grisha Vekstein and Bengt Sonnerup; warm hospitality from and helpful discussions with Terry Forbes and Joe Hollweg in Durham where this work was undertaken and financial support from the UK Particle Physics and Astronomy Research Council and NASA. V.S.T. is grateful to Eric Priest and all his colleagues from the Mathematics Department for their kind hospitality during his visit to St Andrews University.

References

Biskamp, D. 1986 *Phys. Fluids* **29**, 1520–1531.

Bulanov, S. V. & Olshanetsky, M. A. 1984 *Phys. Lett.* A **100**, 35–38.

Bulanov, S. V., Pegoraro, F. & Sakharov, A. S. 1992 *Phys. Fluids* B **4**, 2499–2508.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Cowley, S. 1973 *Radio Science* **8**, 903–913.
Craig, I. J. D. & McClymont, A. N. 1991 *Astrophys. Jl* **371**, L41–L44.
Craig, I. J. D. & Rickard, G. J. 1994 *Astr. Astrophys.* **287**, 261.
Fukao, S., Masayuki, U. & Takao, T. 1975 *Rep. Ion. Space Res. Jap.* **29**, 133–139.
Greene, J. M. 1988 *J. Geophys. Res.* **93**, 8583–8590.
Hesse, M. & Schindler, K. 1988 *J. Geophys. Res.* **93**, 5559–5567.
Jardine, M. & Priest, E. R. 1988 *J. Plasma Phys.* **40**, 143–161.
Jardine, M. & Priest, E. R. 1989 *J. Plasma Phys.* **42**, 111–132.
Jardine, M. & Priest, E. R. 1989 *J. Plasma Phys.* **43**, 141–150.
Lau, Y. T. & Finn, J. M. 1990 *Astrophys. Jl* **350**, 672–691.
Parnell, C., Smith, J., Neukirch, T. & Priest, E. R. 1996 *Phys. Plasmas* **3**, 759–770.
Petschek, H. E. 1964 In *AAS-NASA Symp. on Phys. of Solar Flares*, pp. 425–439. NASA SP-50.
Priest, E. R. & Démoulin, P. 1995 *J. Geophys. Res.* **100**, 23443–23463.
Priest, E. R. & Forbes, T. G. 1986 *J. Geophys. Res.* **91**, 5579–5588.
Priest, E. R. & Forbes, T. G. 1989 *Solar Phys.* **119**, 211–214.
Priest, E. R. & Forbes, T. G. 1992 *J. Geophys. Res.* **97**, 1521–1531.
Priest, E. R. & Lee, L. C. 1990 *J. Plasma Phys.* **44**, 337–360
Priest, E. R., Titov, V. S., Vekstein, G. E. & Rickard, G. J. 1994 *J. Geophys. Res.* **99**, 21467–21480.
Schindler, K., Hesse, M. & Birn, J. 1988 *J. Geophys. Res.* **93**, 5547–5557.
Scholer, M. 1991 *Geophys. Astrophys. Fluid Dyn.* **62**, 51–68.
Stern, D. P. 1973 *J. Geophys. Res.* **78**, 7292–7305.
Strachan, N. R. & Priest, E. R. 1994 *Geophys. Astrophys. Fluid Dyn.* **74**, 245–274.
Titov, V. S. & Priest, F. R. 1996 *Adv. Space Res.* (In the press.)
Yan, M., Lee, L. C. & Priest, E. R. 1992 *J. Geophys. Res.* **97**, 8277–8293.
Yan, M., Lee, L. C. & Priest, E. R. 1993 *J. Geophys. Res* **98**, 7593–7602.

Received 21 February 1995; accepted 11 May 1995

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES